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## ***Lectures on the Theory of Reciprocants.***

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[Reported by J. HAMMOND.]

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The lectures here reproduced were delivered, or are still in the course of delivery, before a class of graduates and scholars in the University of Oxford during the present year. They are to be regarded as easy lessons in the new Theory of Reciprocants of which an outline will be found in *Nature* for January 7, which contains a report of a Public Lecture on the subject delivered before the University of Oxford in December of the preceding year.

They are designed as a practical introduction to an enlarged theory of Algebraical Forms, and as such are not constructed with the rigorous adhesion to logical order which might be properly expected in a systematic treatise. The object of the lecturer was to rouse an interest in the subject, and in pursuit of this end he has not hesitated to state many results, by way of anticipation, which might, with stricter regard to method, have followed at a later point in the course.

There will be found also occasional repetitions and intercalations of allied topics which are to be justified by the same plea, and also by the fact that the lectures were not composed in their entirety previous to delivery, but gradually evolved and written between one lecture and another in the way that seemed most likely to the lecturer to secure the attention of his auditors.

Since the delivery of his public lecture in December last, papers have been contributed on the subject to the Proceedings of the Mathematical Society of London by Messrs. Hammond, MacMahon, Elliott, Leudesdorf and Rogers, and one to the *Comptes Rendus de l'Institut* by M. George Perrin. It may therefore be inferred that the lectures have not altogether failed in attaining the

desired end of drawing attention to a subject which, in the opinion of the lecturer, constitutes a very considerable extension of the previous limits of algebraical science.

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## LECTURE I.

A new world of Algebraical forms, susceptible of important geometrical applications, has recently come into existence, of which I gave a general account in a public lecture at the end of last term. I propose in the following brief course to go more fully into the subject and lay down the leading principles of the theory so far as they are at present known to me. The parallelism between the theory of what may be called pure reciprocants and that of invariants is so remarkable that it will be frequently expedient to pass from one theory to the other or to treat the two simultaneously. It may be as well therefore at once to give notice that the term invariant will hereafter be applied alike to invariants ordinarily so called and to those more general algebraical forms which have been termed sources of covariants, differentiants, seminvariants, or subinvariants. A form which is an invariant in the old sense will be termed, when necessary to specify it, a satisfied invariant, an expression which the chemico-graphical representation of invariants or covariants will serve to explain and justify.

In an elucidatory course of lectures such as the present, it will be advisable to follow a freer order of treatment than would be suitable to the presentation of it in a systematic memoir. My object is to make the theory known, to excite curiosity regarding it, and to invite co-operation in the task of its development.

By way of introduction to the subject, let us begin with an investigation of the properties of a differential expression involving only the first, second and third differential coefficients of either of two variables in respect to the other. For this purpose let us consider not what I have called the Schwarzian itself, which is an integral rational function of these three quantities, but the fractional expression

$$\frac{\frac{d^3y}{dx^3}}{\frac{dy}{dx}} - \frac{3}{2} \left( \frac{\frac{d^2y}{dx^2}}{\frac{dy}{dx}} \right)^2$$

which becomes the Schwarzian when cleared of fractions, and which after Cayley we may call the Schwarzian Derivative and denote by

$$(y, x);$$

$(x, y)$  will then of course mean

$$\frac{\frac{d^3x}{dy^3}}{\frac{dx}{dy}} - \frac{3}{2} \left( \frac{\frac{d^2x}{dy^2}}{\frac{dx}{dy}} \right)^2.$$

It is easy to establish the identical equation

$$(y, x) = - \left( \frac{dy}{dx} \right)^2 (x, y). \quad (1)$$

Using for brevity  $y', y'', y'''$  to denote, as usual,  $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$ ,

and  $x', x'', x'''$  to denote  $\frac{dx}{dy}, \frac{d^2x}{dy^2}, \frac{d^3x}{dy^3}$ , respectively, the relation to be verified is

$$\frac{2y'y''' - 3y''^2}{y'^2} = -y'^2 \cdot \frac{2x'x''' - 3x''^2}{x'^3}.$$

Now,  $x' = \frac{1}{y'}$

$$x'' = \frac{d}{dy} (x') = \frac{1}{y'} \cdot \frac{d}{dx} \left( \frac{1}{y'} \right) = -\frac{y''}{y'^3}$$

and  $x''' = \frac{d}{dy} (x'') = \frac{1}{y'} \cdot \frac{d}{dx} \left( -\frac{y''}{y'^3} \right) = -\frac{y'''}{y'^4} + \frac{3y''^2}{y'^5}.$

Whence we obtain

$$\begin{aligned} 2x'x''' - 3x''^2 &= \left( -\frac{2y'''}{y'^5} + \frac{6y''^2}{y'^6} \right) - \frac{3y''^2}{y'^6} \\ &= -\frac{1}{y'^6} (2y'y''' - 3y''^2), \end{aligned}$$

and the truth of (1) is manifest.

This may be put under the form

$$\frac{2y'y''' - 3y''^2}{y'^3} = -\frac{2x'x''' - 3x''^2}{x'^3},$$

showing that a certain function of the first, second and third derivatives of one variable in respect to another remains unaltered, save as to algebraical sign, when the variables are interchanged. An example of a similar kind with which

we are all familiar is presented by the well-known function  $\frac{d^2y}{dx^2} \div \left( \frac{dy}{dx} \right)^{\frac{3}{2}}$ , which

is equal to  $-\frac{d^2x}{dy^2} \div \left( \frac{dx}{dy} \right)^{\frac{3}{2}}.$

We are thus led to inquire whether there may not be an infinite number of algebraical functions of differential derivatives which possess a similar property, and by prosecuting this inquiry to lay the foundations of the theory of Reciprocation or Reciprocants.

Having regard to the fact that the present theory originated in that of the Schwarzian Derivative, I shall proceed to demonstrate, although this is not strictly necessary for the theory of Reciprocants, the remarkable identity

$$(y, x) - (z, x) = \left(\frac{dz}{dx}\right)^2 (y, z).$$

This identical relation is the fundamental property of Schwarzians, and from it every other proposition concerning their form is an immediate deduction.

In the following proof,\*  $y$  and  $z$  are regarded as two given functions of any variable  $t$ , and  $x$  as a variable function of the same: so that  $y$  and  $z$  are functions of  $x$  for any given function that  $x$  is of  $t$ .

It will be seen that

$$((y, x) - (z, x)) \left(\frac{dx}{dz}\right)^2$$

remains unaltered by any infinitesimal variation  $\theta$  of  $x$ , *i. e.* when  $x$  becomes  $x + \epsilon\phi(x)$ ,  $\epsilon$  being an infinitesimal constant and  $\phi(x)$  an arbitrary finite function of  $x$ .

For brevity, let accents denote differential derivation in regard to  $x$ , and let any function of  $x$  enclosed in a square parenthesis signify the augmented value of that function when  $x$  becomes  $x + \theta$ . In calculating such augmented values, since we suppose that  $\theta = \epsilon\phi(x)$ , it is clear that  $\theta, \theta', \theta'', \dots$  are each of them infinitesimals of the first order, and consequently that all products, and all powers higher than the first of these quantities, may be neglected.

We have therefore

$$\begin{aligned} [y'] &= \frac{dy}{dx + d\theta} = \frac{y'}{1 + \theta'} = y' - \theta'y' \\ [y''] &= \frac{d[y']}{dx + d\theta} = \frac{\frac{d}{dx}(y' - \theta'y')}{1 + \theta'} = \frac{y''(1 - \theta') - \theta''y'}{1 + \theta'} \\ &= y'' - 2\theta'y'' - \theta''y' \\ [y'''] &= \frac{d[y'']}{dx + d\theta} = \frac{\frac{d}{dx}(y'' - 2\theta'y'' - \theta''y')}{1 + \theta'} = \frac{y'''(1 - 2\theta') - 3\theta''y'' - \theta'''y'}{1 + \theta'} \\ &= y''' - 3\theta'y''' - 3\theta''y'' - \theta'''y'. \end{aligned}$$

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\* As originally given in the *Messenger of Mathematics* (Vol. XV,      ), this was defaced by so many errata as to render expedient its reproduction in a corrected form.

Hence

$$\begin{aligned} [y'y'''] &= y'y''' - 4\theta'y'y''' - 3\theta''y'y'' - \theta'''y'^2 \\ \frac{3}{2}[y''^2] &= \frac{3}{2}y''^2 - 6\theta'y''^2 - 3\theta''y'y'' \\ [y'^2] &= y'^2 - 2\theta'y'^2. \end{aligned}$$

And since by definition

$$(y, x) = \frac{y'y''' - \frac{3}{2}y''^2}{y'^2},$$

we readily obtain

$$[(y, x)] = \frac{(y, x)}{1 - 2\theta'} - 4\theta'(y, x) - \theta''' = (y, x)(1 - 2\theta') - \theta''.$$

So also

$$[(z, x)] = (z, x)(1 - 2\theta') - \theta''.$$

Whence by subtraction

$$[(y, x) - (z, x)] = (1 - 2\theta')\{(y, x) - (z, x)\}.$$

Dividing the left-hand side of this by  $[z'^2]$ , and the right-hand side by  $z'^2(1 - 2\theta')$  which is the equivalent of  $[z'^2]$ , our final result is

$$\left[ \frac{(y, x) - (z, x)}{z'^2} \right] = \frac{(y, x) - (z, x)}{z'^2}.$$

Thus, then, we have seen that the expression

$$\frac{(y, x) - (z, x)}{\left(\frac{dz}{dx}\right)^2}$$

does not vary when  $x$  receives an infinitesimal variation  $\varepsilon\phi(x)$ , from which it follows, by the general principle of successive continuous accumulation, that the same will be true when  $x$  undergoes any finite arbitrary variation, and consequently this expression has a value which is independent of the form of  $x$  regarded as a function of  $t$ ; it will, of course, be remembered that  $y$  and  $z$  are supposed to be invariable functions of  $t$ . Let  $x$  become  $z$ , then  $(y, x)$  becomes  $(y, z)$ , while at the same time  $(z, x)$  vanishes and  $\frac{dz}{dx}$  becomes unity: so that we obtain

$$\frac{(y, x) - (z, x)}{\left(\frac{dz}{dx}\right)^2} = (y, z).$$

Hence, *whatever* function  $x$  may be of  $t$ ,

$$(y, x) - (z, x) = \left(\frac{dz}{dx}\right)^2 \cdot (y, z). \quad (2)$$

To this fundamental proposition the equation marked (1), itself the important

point in regard to the Theory of Reciprocants, is an immediate corollary. For if in (2) we interchange  $y$  and  $z$ , it becomes

$$(z, x) - (y, x) = \left(\frac{dy}{dx}\right)^2 \cdot (z, y);$$

and now, making  $x = z$ , we have

$$- (y, z) = \left(\frac{dy}{dz}\right)^2 \cdot (z, y),$$

which is the same as (1), except that  $z$  occupies the place of  $x$ .

But (1) may be obtained more immediately from (2) by substituting in it  $x$  for  $y$  and  $y$  for  $z$ , leaving  $x$  unaltered; when it becomes

$$- (y, x) = \left(\frac{dy}{dx}\right)^2 \cdot (x, y).$$

This is equivalent to saying that

$$2y'y''' - 3y''^2 = -y'^6(2x_1x_{111} - 3x_{11}^2),$$

a verification of which has been given already.

Observe that  $\frac{y'y''' - \frac{3}{2}y''^2}{y'^2}$  or  $(y, x)$  contains  $\left(\frac{dy}{dx}\right)^2$  in its denominator and  $(x, y)$  contains  $\left(\frac{dx}{dy}\right)^2$  in its denominator, which is the same as  $\left(\frac{dy}{dx}\right)^2$  in the numerator. Thus it is that the *square* of  $\frac{dy}{dx}$  enters three times.

Let me insist for a moment on the import of the fact brought to light in the course of this investigation, that  $\frac{(y, x) - (z, x)}{\left(\frac{dz}{dx}\right)^2}$  is invariable when  $x, y$  and  $z$  being regarded as functions of  $t$ ,  $x$  alters its form, but  $y$  and  $z$  retain theirs. Of course we might write  $\left(\frac{dy}{dx}\right)^2$  in the denominator instead of  $\left(\frac{dz}{dx}\right)^2$ , and then make the same affirmation as before; as will be evident if we only remember that by hypothesis  $y$  and  $z$  are both of them constant functions of  $t$ , and that therefore  $\left(\frac{dz}{dy}\right)^2$  must also be so. This is tantamount to saying that when the same conditions are fulfilled  $((y, x) - (z, x))(dx)^2$  is invariable, *i. e.* that when  $x$  becomes  $X$  in virtue of any substitution (including a homographic one) impressed upon it,

$$\{(y, x) - (z, x)\}(dx)^2 = \{(y, X) - (z, X)\}(dX)^2,$$

and thus we see that when  $x$  becomes  $X$ ,

$$(y, x) - (z, x)$$

remains unaltered except that it takes to itself the factor  $\left(\frac{dX}{dx}\right)^2$  which depends solely on the particular substitution impressed on  $x$ .

If  $y = f(x)$ ,  $z = \phi(x)$ , and  $X = \omega(x)$ ,  
our formula becomes

$$\{(fx, x) - (\phi x, x)\}(dx)^2 = \{(f\omega^{-1}X, X) - (\phi\omega^{-1}X, X)\}(dX)^2,$$

so that, speaking of Quantics and Covariants with respect to a single variable  $x$ ,  $(fx, x) - (\phi x, x)$  is to all intents and purposes a Covariant to the simultaneous forms  $f(x)$  and  $\phi(x)$ , in a sense comprehending but far transcending that in which the term is ordinarily employed; for it remains a persistent factor of its altered self when for  $x$  any arbitrary function of  $x$  is substituted, the new factor taken on depending wholly and solely on the particular substitution impressed upon  $x$ . In the ordinary theory of invariants, the substitution impressed is limited to be homographic; in this case it is absolutely general. We might, moreover, add as a corollary that if  $(y, x)$ ,  $(z, x)$ ,  $(u, x)$  . . . are regarded as roots of any Binary Quantic, every invariant of that Binary Quantic is a covariant in the extended sense in which the word has just been used, in respect to the system of simultaneous forms  $f(x)$ ,  $\phi(x)$ ,  $\psi(x)$  . . . . For every such invariant will be a function of  $(y, x) - (z, x)$ ,  $(y, x) - (u, x)$ ,  $(z, x) - (u, x)$ , . . . and will therefore remain a persistent factor of its altered self, taking on a power of  $\frac{dX}{dx}$  as its extraneous factor.

Calling  $(fx, x)$  the Schwarzian Derivative of  $f(x)$ , our theorem may be stated in general terms as follows:

*All invariants of a Binary Quantic whose roots are the Schwarzian Derivatives of a given set of functions of the same variable are Covariants (in an extended sense) of that set of functions.*

The theory of the Schwarzian derivative originates in that of the linear differential equation of the second order,

$$w'' + 2Pu' + Qu = 0,$$

which becomes, when we write  $u = ve^{-\int P dx}$ ,

$$v'' + Iv = 0,$$

where

$$I = Q - P^2 - P'.$$

Now, suppose that  $u_1$  and  $u_2$  are any two particular solutions of the first of these

equations, and let  $z$  denote their mutual ratio; so that, when  $v_1$  and  $v_2$  are the corresponding particular solutions of the second equation, we readily obtain

$$z = \frac{v_2}{v_1} = \frac{v_2}{v_1},$$

and therefore,

$$z' = \frac{v_1 v_2' - v_2 v_1'}{v_1^2}.$$

A second differentiation gives

$$z'' = \frac{v_1 v_2'' - v_2 v_1''}{v_1^3} - \frac{2v_1'(v_1 v_2' - v_2 v_1')}{v_1^3}.$$

But since

$$\frac{v_1''}{v_1} = \frac{v_2''}{v_2} = -I,$$

the first term of the expression just found vanishes identically, and we have

$$z'' = -\frac{2v_1' z'}{v_1},$$

or,

$$v_1' = -\frac{z'' v_1}{2z'}.$$

Differentiating this again, we find

$$\begin{aligned} -2v_1'' &= \left( \frac{z'''}{z'} - \frac{z''^2}{z'^2} \right) v_1 + \frac{z''}{z'} v_1' \\ &= \left( \frac{z'''}{z'} - \frac{3}{2} \frac{z''^2}{z'^2} \right) v_1. \end{aligned}$$

Hence

$$\frac{z'''}{z'} - \frac{3}{2} \frac{z''^2}{z'^2} = 2I,$$

where the left-hand side of the equation is “the Schwarzian Derivative” with  $z$  written in the place of  $y$ .

## LECTURE II.

The expression  $2y'y''' - 3y''^2$ , which we have called the Schwarzian, may be termed a reciprocant, meaning thereby that on interchanging  $y'$ ,  $y''$ ,  $y'''$  with  $x'$ ,  $x''$ ,  $x'''$  its form remains unaltered, save as to the acquisition of what may be called an extraneous factor, which, in the case before us, is a power of  $y'$  (with a multiplier  $-1$ ). Before we proceed to consider other examples of reciprocants it will be useful to give formulae by means of which the variables may be readily interchanged in any differential expression.

We shall write  $t$  for  $y'$  and  $\tau$  for its reciprocal  $x$ , using the letters  $a, b, c, \dots$  to denote the second, third, fourth, etc., differential derivatives of  $y$  with respect to  $x$ , and  $\alpha, \beta, \gamma, \dots$  to denote those of  $x$  with respect to  $y$ . The advantage of this notation will be seen in the sequel.

The values of  $\alpha, \beta, \gamma, \dots$  in terms of  $t, a, b, c, \dots$  are given by the formulae

$$\begin{aligned}\alpha &= -a \div t^3, \\ \beta &= -bt + 3a^2 \div t^5, \\ \gamma &= -ct^2 + 10abt - 15a^3 \div t^7, \\ \delta &= -dt^3 + (15ac + 10b^2)t^2 - 105a^2bt + 105a^4 \div t^9, \\ \varepsilon &= -et^4 + (21ad + 35bc)t^3 - (210a^2c + 280ab^2)t^2 + 1260a^3bt - 945a^5 \div t^{11}, \\ &\dots\end{aligned}$$

If, in these equations, we write  $a = 1.2.a_0$ ,  $b = 1.2.3.a_1$ ,  $c = 1.2.3.4.a_2$ ,  $\dots$  and  $\alpha = 1.2.\alpha_0$ ,  $\beta = 1.2.3.\alpha_1$ ,  $\gamma = 1.2.3.4.\alpha_2$ ,  $\dots$  they become

$$\begin{aligned}\alpha_0 &= -a_0 \div t^3, \\ \alpha_1 &= -a_1t + 2a_0^2 \div t^5, \\ \alpha_2 &= -a_2t^2 + 5a_0a_1t - 5a_0^3 \div t^7, \\ \alpha_3 &= -a_3t^3 + (6a_0a_2 + 3a_1^2)t^2 - 21a_0^2a_1t + 14a_0^4 \div t^9, \\ \alpha_4 &= -a_4t^4 + (7a_0a_3 + 7a_1a_2)t^3 - (28a_0^2a_2 + 28a_1^2)t^2 + 84a_0^3a_1t - 42a_0^5 \div t^{11}, \\ &\dots\end{aligned}$$

Any one of the formulae in either set may be deduced from the formula immediately preceding it by a simple process of differentiation.

Thus, since  $\beta = \frac{-bt + 3a^2}{t^5}$  and  $\frac{d}{dy} = \frac{1}{t} \cdot \frac{d}{dx}$ ,

we have  $\frac{d\beta}{dy} = \frac{1}{t} \cdot \frac{d}{dx} \left( \frac{-bt + 3a^2}{t^5} \right)$ .

But  $\frac{d\beta}{dy} = \gamma$  and  $\frac{d}{dx} = a\partial_t + b\partial_a + c\partial_b + \dots$ ,

so that 
$$\begin{aligned}\gamma &= \frac{1}{t} (a\partial_t + b\partial_a + c\partial_b + \dots) \left( \frac{-bt + 3a^2}{t^5} \right) \\ &= \frac{1}{t^4} (-ct^2 + 10abt - 15a^3).\end{aligned}$$

By continually operating with  $\frac{1}{t} (a\partial_t + b\partial_a + c\partial_b + \dots)$  the table may be extended as far as we please, the expressions on the right-hand side being the successive values of

$$\left\{ \frac{1}{t} (a\partial_t + b\partial_a + c\partial_b + \dots) \right\}^n \left( -\frac{a}{t^3} \right)$$

found by giving to  $n$  the values  $0, 1, 2, 3, \dots$

Precisely similar reasoning shows that, when the modified letters  $a_0, a_1, a_2, \dots$  are used,

$$(n+2)\alpha_n = \frac{1}{t} (2a_0\partial_t + 3a_1\partial_{a_0} + 4a_2\partial_{a_1} + \dots)\alpha_{n-1},$$

and that

$$\alpha_n = \frac{\left\{ \frac{1}{t} (2a_0\partial_t + 3a_1\partial_{a_0} + 4a_2\partial_{a_1} + \dots) \right\}^n \left( -\frac{a_0}{t^3} \right)}{3.4.5 \dots (n+2)}.$$

A proof of the formula

$$\alpha_n = -t^{-n-3} (e^{-\frac{v}{t}}) \alpha_n,$$

obtained by Mr. Hammond, in which

$$V = 4 \cdot \frac{a_0^2}{2} \partial_{a_1} + 5a_0a_1\partial_{a_2} + 6 \left( a_0a_2 + \frac{a_1^2}{2} \right) \partial_{a_3} + 7(a_0a_3 + a_1a_2) \partial_{a_4} + \dots$$

will be given later on, when we treat of this operator, which, in the theory of Reciprocants, is the analogue of the operator  $a\partial_b + 2b\partial_c + 3c\partial_d + \dots$  with which we are familiarly acquainted in the theory of Invariants.

Consider the expression  $ct - 5ab$ .

If, in  $\gamma\tau - 5\alpha\beta$ , which may be called its transform, we write

$$\tau = \frac{1}{t}, \quad \alpha = -\frac{a}{t^3}, \quad \beta = \frac{-bt + 3a^2}{t^5}, \quad \gamma = \frac{-ct^2 + 10abt - 15a^3}{t^7},$$

this becomes a fraction whose denominator is  $t^8$ , while its numerator is

$$-ct^2 + 10abt - 15a^3 + 5a(-bt + 3a^2) = -ct^2 + 5abt.$$

Removing the common factor  $t$  from the numerator and denominator of this fraction, we have

$$\gamma\tau - 5\alpha\beta = -\frac{ct - 5ab}{t^7}.$$

Here, then, as in the case of the well-known monomial for which

$$a = -t^3\alpha,$$

and the Schwarzian for which

$$2bt - 3a^2 = -t^6(2\beta\tau - 3\alpha^2),$$

the expression

$$ct - 5ab = -t^7(\gamma\tau - 5\alpha\beta)$$

changes its sign on reciprocation.

That reciprocation is not always accompanied with a change of sign will be clear if we consider the product of any pair of the three expressions given above. Or we may take, as an example of a reciprocant in which this change of sign does not occur, the form

$$3ac - 5b^2.$$

Here

$$3a\gamma - 5\beta^2 = \frac{3a(ct^2 - 10abt + 15a^3) - 5(bt - 3a^2)^2}{t^{10}}.$$

In the fraction on the right-hand side the only surviving terms of the numerator are those containing the highest power of  $t$ , the rest destroying one another.

Thus 
$$3\alpha\gamma - 5\beta^2 = \frac{1}{t^6}(3ac - 5b^2).$$

Reciprocants which change their sign when the variables  $x$  and  $y$  are interchanged, will be said to be of odd character; those, on the contrary, which keep their sign unchanged will be said to be of even character. The distinction is an important one, and will be observed in what follows.

Forms such as the one just considered, where  $t$  does not appear in the form itself, but only in the extraneous factor, will be called Pure Reciprocants, in order to distinguish them from those forms (of which the Schwarzian  $2tb - 3a^2$  is an example) into which  $t$  enters, which will be called Mixed Reciprocants. It will be seen hereafter that Pure Reciprocants are the analogues of the invariants of Binary Quantics.

With modified letters (*i. e.* writing  $a = 2a_0$ ,  $b = 6a_1$ , and  $c = 24a_2$ )

$$3ac - 5b^2 \text{ becomes } 144a_0a_2 - 180a_1^2 = 36(4a_0a_2 - 5a_1^2).$$

Operating on this with

$$V = 2a^2\partial_{a_1} + 5a_0a_1\partial_{a_2} + \dots,$$

we have

$$V(4a_0a_2 - 5a_1^2) = 0.$$

We shall prove subsequently that all Pure Reciprocants are, in like manner, subject to annihilation by the operator  $V$ .

Hitherto we have only considered homogeneous; let us now take as an example of a non-homogeneous reciprocant the expression

$$(1 + t^2)b - 3a^2t.$$

Here 
$$\begin{aligned} (1 + \tau^2)\beta - 3\alpha^2\tau &= \left(1 + \frac{1}{t^2}\right)\left(\frac{-bt + 3a^2}{t^5}\right) - \frac{3a^2}{t^7} \\ &= \frac{(1 + t^2)(-bt + 3a^2) - 3a^2}{t^7}. \end{aligned}$$

In the numerator of this fraction the terms  $+3a^2$  and  $-3a^2$  cancel, a factor  $t$  divides out, and we have finally

$$(1 + \tau^2)\beta - 3\alpha^2\tau = -\frac{(1 + t^2)b - 3a^2t}{t^6}.$$

In general, a Reciprocant may be defined to be a function  $F$  of such a kind that  $F(\tau, \alpha, \beta, \gamma, \dots)$  contains  $F(t, a, b, c, \dots)$  as a factor. An important special case is that in which the other factor is merely numerical; the function  $F$  is then said to be an Absolute Reciprocant.

When we limit ourselves to the case where  $F$  is a rational integral function of the letters, it may be proved that

$$F(t, a, b, c, \dots) = \pm t^u F(\tau, \alpha, \beta, \gamma, \dots).$$

For, in the first place, since any one of the letters  $\alpha, \beta, \gamma, \dots$  is a rational function of  $t, a, b, c, \dots$  and integral with respect to all of them except  $t$ , containing only a power of this letter in the denominator, it is clear that any rational integral function of  $\tau, \alpha, \beta, \gamma, \dots$  such as  $F(\tau, \alpha, \beta, \gamma, \dots)$  is supposed to be, must be a rational integral function of  $t, a, b, c, \dots$  divided by some power of  $t$ . But since  $F$  is a reciprocant,  $F(\tau, \alpha, \beta, \gamma, \dots)$  must contain  $F(t, a, b, c, \dots)$  as a factor; and if we suppose the other factor to be  $\frac{\phi(t, a, b, c, \dots)}{t^\lambda}$  we must have

$$F(\tau, \alpha, \beta, \gamma, \dots) = \frac{\phi(t, a, b, c, \dots)}{t^\lambda} F(t, a, b, c, \dots),$$

where  $\phi$  is rational and integral with respect to all the letters.

Moreover,

$$F(t, a, b, c, \dots) = \frac{\phi(\tau, \alpha, \beta, \gamma, \dots)}{\tau^\lambda} F(\tau, \alpha, \beta, \gamma, \dots).$$

Hence we must have identically

$$\phi(t, a, b, c, \dots) \phi(\tau, \alpha, \beta, \gamma, \dots) = 1,$$

where, on the supposition that the functions  $\phi$  contain other letters besides  $t$  and  $\tau$ ,  $\phi(t, a, b, c, \dots)$  is, and  $\phi(\tau, \alpha, \beta, \gamma, \dots)$  can be expressed as, a rational function integral as regards the letters  $a, b, c, \dots$ . But this supposition is manifestly inadmissible, for the product of two integral rational functions of  $a, b, c, \dots$  cannot be identically equal to unity. Hence  $t$  is the only letter that can appear in the extraneous factor and we may write

$$F(\tau, \alpha, \beta, \gamma, \dots) = \frac{\psi(t)}{t^\lambda} F(t, a, b, c, \dots)$$

where  $\psi(t)$  is a rational integral function.

The same reasoning as before shows that we must have identically

$$\psi(t) \psi(\tau) = 1.$$

But this cannot be true if  $\psi(t)$  has any root different from zero; for if we give  $t$  such a value as will make  $\psi(t)$  vanish, this value must also make  $\psi(\tau)$  infinite; and since

$$\begin{aligned} \psi(\tau) &= A + B\tau + C\tau^2 + \dots + M\tau^m \\ &= A + \frac{B}{t} + \frac{C}{t^2} + \dots + \frac{M}{t^m}, \end{aligned}$$

the only value of  $t$  for which  $\psi(\tau)$  becomes infinite is a zero value. Hence  $\psi(t)$  is of the form  $Mt^m$ , and consequently  $\psi(\tau) = M\tau^m$ . Thus

$$\psi(t)\psi(\tau) = M^2 t^m \tau^m = 1,$$

and therefore

$$M^2 = 1.$$

We have now proved that if  $F$  is a rational integral reciprocant,

$$F(t, a, b, c, \dots) = \pm t^\mu F(\tau, \alpha, \beta, \gamma, \dots),$$

or we may say,

$$= (-)^{\kappa t^\mu} F(\tau, \alpha, \beta, \gamma, \dots),$$

where  $\kappa = 1$  or  $0$  according as the reciprocant is of odd or even character.

It obviously follows that the product or quotient of any two rational integral reciprocants is itself a reciprocant; but it must be carefully observed that this is not true of their sum or difference unless certain conditions are fulfilled. For if we write

$$F_1(t, a, \dots) = (-)^{\kappa_1 t^{\mu_1}} F_1(\tau, \alpha, \dots)$$

and

$$F_2(t, a, \dots) = (-)^{\kappa_2 t^{\mu_2}} F_2(\tau, \alpha, \dots),$$

we see that

$$pF_1(t, a, \dots) + qF_2(t, a, \dots) = (-)^{\kappa_1 t^{\mu_1}} pF_1(\tau, \alpha, \dots) + (-)^{\kappa_2 t^{\mu_2}} qF_2(\tau, \alpha, \dots),$$

and consequently this expression will be a reciprocant if  $\kappa_1 = \kappa_2$  and  $\mu_1 = \mu_2$ , but not otherwise. If we call the index of  $t$  in the extraneous factor the *characteristic*, what we have proved is that no linear function of two reciprocants can be a reciprocant, unless they have the same characteristic and are of the same character. In dealing with Absolute Reciprocants, since the characteristic of these is always zero, we need only attend to their character.

I propose for the present to confine myself to homogeneous and isobaric reciprocants,\* *i. e.* to such as are homogeneous and isobaric when the letters  $t, a, b, c, \dots$  are considered to be each of degree 1, their respective weights being  $-1, 0, 1, 2, \dots$ . The letter  $w$  will be used to denote the weight of such a reciprocant,  $i$  its degree, and  $j$  its extent, *i. e.* the weight of the most advanced letter which it contains.

Let any such reciprocant  $F(t, a, b, c, \dots)$  contain a term  $A t^v a^l b^m c^n \dots$ , then

$$v + l + m + n + \dots = i$$

and

$$-v + m + 2n + \dots = w.$$

The corresponding term in  $F(\tau, \alpha, \beta, \gamma, \dots)$  will be  $A \tau^v \alpha^l \beta^m \gamma^n \dots$  where

$$\tau = \frac{1}{t}, \quad \alpha = -\frac{a}{t^2}, \quad \beta = -\frac{b}{t^4} + \dots, \quad \gamma = -\frac{c}{t^6} + \dots, \text{ etc.}$$

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\* Here and elsewhere the word *reciprocant* is used in the sense of *rational integral reciprocant*: this will always be done when there is no danger of confusion arising from it.

Now, if no term of  $F$  contains a smaller number of the letters  $a, b, c, \dots$  than are found in the term we are considering, the first terms of  $\beta, \gamma$ , etc., may be taken instead of these quantities themselves and  $A\tau^v\alpha^l\beta^m\gamma^n\dots$  may be replaced by

$$(-)^{i+m+n+\dots}At^{-v-3l-4m-5n-\dots}a^lb^mc^n\dots = (-)^{i-v}At^{v-3i-w}a^lb^mc^n\dots$$

But since

$$F(t, a, b, c, \dots) = (-)^{\kappa t^{\mu}}F(\tau, \alpha, \beta, \gamma, \dots)$$

we must have identically

$$At^va^lb^mc^n\dots = (-)^{i-v+\kappa}At^{\mu+v-3i-w}a^lb^mc^n\dots$$

Hence the character is even or odd according to the parity of  $i - v$  (*i. e.* of the smallest number of letters different from  $t$  in any term), and the characteristic  $\mu = 3i + w$ .

The type of a reciprocant depends on the *character*, weight, degree and extent. As the extraneous factor is always of the form  $(-)^{\kappa t^{\mu}}$ , where  $\kappa$  is 1 or 0, we may define the type of a reciprocant by

$$1:w:i,j \quad \text{or} \quad 0:w:i,j,$$

according as its character is odd or even.

For Pure Reciprocants the smallest number of letters different from  $t$  in any term is (since all the letters are different from  $t$ ) the same as its degree. Hence the character of a Pure Reciprocant is odd or even according to the parity of  $i$ , and for this reason the type of a Pure Reciprocant may be defined by

$$w:i,j.$$

A linear combination of reciprocants of the same type will be a reciprocant, for when the type is known both the character and characteristic are given.

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### LECTURE III.

Let  $F$  be any function (not necessarily homogeneous or even algebraical) of the differential derivatives which acquires a numerical multiplier  $M$ , but is otherwise unchanged when the reciprocal substitution of  $x$  for  $y$  and  $y$  for  $x$  is effected. A second reciprocation multiplies the function again by  $M$ , and thus the total effect of both substitutions is to multiply  $F$  by  $M^2$ . But since the

second reciprocation reproduces the original function, we must have  $M^2 = 1$ . Functions of this kind are therefore unaltered by reciprocation (except it may be in sign), and for this reason are called *Absolute Reciprocants*. These, as we shall presently see, play an important part in the general theory. Like all other reciprocants, they range naturally in two distinct classes, those of odd and those of even character.

It is perhaps worthy of notice that the extraneous factor of a general reciprocant is the exponential of an absolute reciprocant of odd character. For if

$$F(t, a, b, c, \dots) = \phi(t, a, b, c, \dots) F(\tau, \alpha, \beta, \gamma, \dots),$$

we must still have, as before,

$$\phi(t, a, b, c, \dots) \phi(\tau, \alpha, \beta, \gamma, \dots) = 1;$$

i. e.  $\log \phi(t, a, b, c, \dots) = -\log \phi(\tau, \alpha, \beta, \gamma, \dots);$

or  $\log \phi(t, a, b, c, \dots)$  is an absolute reciprocant of odd character.

An absolute reciprocant may be obtained from any pair of rational integral reciprocants in the same way that an absolute invariant is found from two ordinary invariants. For let

$$F_1(t, a, b, c, \dots) = (-)^{\kappa_1 t^{\mu_1}} F_1(\tau, \alpha, \beta, \gamma, \dots),$$

and 
$$F_2(t, a, b, c, \dots) = (-)^{\kappa_2 t^{\mu_2}} F_2(\tau, \alpha, \beta, \gamma, \dots),$$

then 
$$\frac{\{F_1(t, a, b, c, \dots)\}^{\mu_2}}{\{F_2(t, a, b, c, \dots)\}^{\mu_1}} = (-)^{\kappa_1 \mu_2 - \kappa_2 \mu_1} \frac{\{F_1(\tau, \alpha, \beta, \gamma, \dots)\}^{\mu_2}}{\{F_2(\tau, \alpha, \beta, \gamma, \dots)\}^{\mu_1}};$$

or we may say that  $F_1^{\mu_2} \div F_2^{\mu_1}$  is an absolute reciprocant of even or odd character according to the parity of  $\kappa_1 \mu_2 - \kappa_2 \mu_1$ .

Thus, for example, from

$$a = -t^3 \alpha$$

and

$$3ac - 5b^3 = t^8 (3\alpha\gamma - 5\beta^3)$$

we form  $\frac{(3ac - 5b^3)^3}{a^8}$ , an absolute reciprocant of even character.

From a reciprocant  $F$  whose characteristic is  $\mu$  we obtain an absolute reciprocant of the same character as  $F$  by dividing it by  $t^{\frac{\mu}{2}}$ .

For if we only remember that  $\tau = \frac{1}{t}$ , it obviously follows that

$$F(t, a, b, c, \dots) = \pm t^\mu F(\tau, \alpha, \beta, \gamma, \dots)$$

can be written in the form

$$\frac{F(t, a, b, c, \dots)}{t^{\frac{\mu}{2}}} = \pm \frac{F(\tau, \alpha, \beta, \gamma, \dots)}{\tau^{\frac{\mu}{2}}},$$

where the original character of the reciprocant  $F$  is preserved.

It may be noticed that a reciprocant of odd character cannot be divided by  $\sqrt{-1}t^{\frac{\mu}{2}}$  so as to give an absolute reciprocant of even character; for, the reciprocal of  $F$  being  $-t^{\mu}F'$ , that of  $F \div \sqrt{-1}t^{\frac{\mu}{2}}$  will still be  $-F' \div \sqrt{-1}t^{\frac{\mu}{2}}$ . The character of a reciprocant is thus seen to be one of its indelible attributes.

As simple examples of absolute reciprocants we may take  $\frac{3ac-5b^2}{t^4}$ , which becomes on reciprocation  $\frac{3a\gamma-5\beta^2}{\tau^4}$ , and  $\frac{a}{t^{\frac{3}{2}}}$ , which reciprocates into  $-\frac{\alpha}{\tau^{\frac{3}{2}}}$ . The character of the former is even, that of the latter odd.

Observing that

$$\log t = -\log \tau \text{ and } \frac{1}{\sqrt{t}} \cdot \frac{d}{dx} = \frac{1}{\sqrt{\tau}} \cdot \frac{d}{dy},$$

we have 
$$\left(\frac{1}{\sqrt{t}} \cdot \frac{d}{dx}\right) \log t = -\left(\frac{1}{\sqrt{\tau}} \cdot \frac{d}{dy}\right) \log \tau.$$

From this, in like manner, we obtain

$$\left(\frac{1}{\sqrt{t}} \cdot \frac{d}{dx}\right)^2 \log t = -\left(\frac{1}{\sqrt{\tau}} \cdot \frac{d}{dy}\right)^2 \log \tau;$$

and so, in general,

$$\left(\frac{1}{\sqrt{t}} \cdot \frac{d}{dx}\right)^i \log t = -\left(\frac{1}{\sqrt{\tau}} \cdot \frac{d}{dy}\right)^i \log \tau.$$

Hence  $\left(\frac{1}{\sqrt{t}} \cdot \frac{d}{dx}\right)^i \log t$  is an absolute reciprocant, and of an odd character, for all positive integral values of  $i$ . We thus obtain a series of fractions with rational integral homogeneous reciprocants in their numerators and powers of  $t^{\frac{3}{2}}$  in their denominators. It will be sufficient, before proceeding to the more general theory of *Eduction*, as it may be called, to examine, by way of illustration, the cases in which  $i = 1, 2$  and  $3$ .

Let  $i = 1$ ; then

$$\left(\frac{1}{\sqrt{t}} \cdot \frac{d}{dx}\right) \log t = \frac{a}{t^{\frac{3}{2}}}.$$

So that, in the case where  $i = 2$ , we have

$$\begin{aligned} \left(\frac{1}{\sqrt{t}} \cdot \frac{d}{dx}\right)^2 \log t &= \left(\frac{1}{\sqrt{t}} \cdot \frac{d}{dx}\right) \frac{a}{t^{\frac{3}{2}}} = \frac{b}{t^2} - \frac{3}{2} \cdot \frac{a^2}{t^3} \\ &= \frac{2bt - 3a^2}{2t^3}. \end{aligned}$$

The numerator of this fraction is the Schwarzian.

In like manner, when  $i = 3$ ,

$$\left(\frac{1}{\sqrt{t}} \cdot \frac{d}{dx}\right)^3 \log t = \left(\frac{1}{\sqrt{t}} \cdot \frac{d}{dx}\right) \left(\frac{2bt - 3a^2}{2t^3}\right) = \frac{2ct - 4ab}{2t^{\frac{7}{2}}} - \frac{6abt - 9a^3}{t^{\frac{9}{2}}} = \frac{2ct^2 - 10abt + 9a^3}{2t^{\frac{9}{2}}}.$$

But here a reduction may be effected, for  $\left(\frac{a}{t^{\frac{3}{2}}}\right)^3$ , as well as  $\frac{a}{t^{\frac{3}{2}}}$  itself, is an absolute reciprocant of the same character as the whole of the expression just found. Hence we may reject the term  $\frac{9}{2} \cdot \frac{a^3}{t^{\frac{9}{2}}}$  without thereby affecting the reciprocal property of the form, and thus obtain

$$\frac{ct - 5ab}{t^{\frac{7}{2}}},$$

an absolute reciprocant of odd character. The corresponding rational integral reciprocant is

$$ct - 5ab.$$

We have found that  $\frac{a}{t^{\frac{3}{2}}}$  and  $\frac{2bt - 3a^2}{t^3}$  are each of them reciprocants. Why, then, by parity of reasoning, is not  $\frac{2bt}{t^3}$ , and therefore  $b$ , a reciprocant? It is because  $\frac{a^2}{t^3}$ , the square of  $\frac{a}{t^{\frac{3}{2}}}$ , is of even character, while  $\frac{2bt - 3a^2}{t^3}$  is of an odd character, so that no linear combination of the two would be *legitimate*.

If we differentiate any absolute reciprocant with respect to  $x$ , we shall obtain another reciprocant of the same character. For let  $R$  be any absolute reciprocant and  $R'$  its transform, then

$$R = \pm R';$$

and since  $\frac{d}{dx} = t \frac{d}{dy}$  may be written in the equivalent but more symmetrical form

$$\frac{1}{\sqrt{t}} \cdot \frac{d}{dx} = \frac{1}{\sqrt{\tau}} \cdot \frac{d}{dy},$$

we have

$$\left(\frac{1}{\sqrt{t}} \cdot \frac{d}{dx}\right) R = \pm \left(\frac{1}{\sqrt{\tau}} \cdot \frac{d}{dy}\right) R'.$$

On one side of this identical equation is a function of the differential derivatives of  $y$  with respect to  $x$ ; on the other, a precisely similar function of those of  $x$  with respect to  $y$ . Hence  $\frac{1}{\sqrt{t}} \cdot \frac{dR}{dx}$  is an absolute reciprocant, and therefore  $\frac{dR}{dx}$  is a reciprocant, the character of each being the same as that of  $R$ .

I will avail myself of the conclusion just obtained, which is the cardinal property of absolute reciprocants, to give a general method of generating from

any given Rational Integral Reciprocant an infinity of others—rational integral educts of it, we may say. Let  $F$  be such a reciprocant, and  $\mu$  its characteristic; then  $\frac{F}{t^{\frac{\mu}{2}}}$  is an absolute reciprocant, and consequently  $\frac{d}{dx} \left( \frac{F}{t^{\frac{\mu}{2}}} \right)$  is a reciprocant, both of them of the same character as  $F$ ; *i. e.*

$$\frac{t \frac{dF}{dx} - \frac{\mu}{2} \cdot aF}{t^{\frac{\mu}{2}+1}};$$

or we may say

$$2t \frac{dF}{dx} - \mu aF$$

is a reciprocant of the same character as  $F$ .

This is even true for non-homogeneous reciprocants, for the only assumption made at present as to the nature of  $F$  is that it is a rational integral reciprocant. But if we further assume that it is homogeneous and isobaric,\* we know that

$$\mu = 3i + w.$$

Now, Euler's equation gives

$$3i = 3(t\partial_t + a\partial_a + b\partial_b + c\partial_c + \dots),$$

and from the similar equation for isobaric functions (remembering that the weights of  $t, a, b, c, \dots$  are  $-1, 0, 1, 2, \dots$ ) we obtain

$$w = -t\partial_t + b\partial_b + 2c\partial_c + \dots,$$

so that

$$\mu = 2t\partial_t + 3a\partial_a + 4b\partial_b + 5c\partial_c + \dots$$

And since

$$\frac{d}{dx} = a\partial_t + b\partial_a + c\partial_b + d\partial_c + \dots,$$

we may in  $\left( 2t \frac{d}{dx} - \mu a \right) F$  replace  $2t \frac{d}{dx} - \mu a$  by

$$2t(a\partial_t + b\partial_a + c\partial_b + d\partial_c + \dots) \\ - a(2t\partial_t + 3a\partial_a + 4b\partial_b + 5c\partial_c + \dots),$$

or by its equivalent

$$(2bt - 3a^2)\partial_a + (2ct - 4ab)\partial_b + (2dt - 5ac)\partial_c + \dots$$

The conclusion arrived at is that when  $F$  is a rational integral homogeneous reciprocant,

$$\{(2bt - 3a^2)\partial_a + (2ct - 4ab)\partial_b + (2dt - 5ac)\partial_c + \dots\} F$$

is another, and that both are of the same character.

It will be convenient to use the letter  $G$  to denote the operator just found and to speak of it as the generator for mixed reciprocants. By the repeated

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\* It will subsequently be proved that every rational integral reciprocant which is homogeneous is also isobaric.

operation of this generator on  $a$  we may obtain the series  $Ga, G^2a, G^3a, \dots$ , whose terms will be mixed reciprocants, since each operation increases the highest power of  $t$  by unity. The forms thus obtained will, in general, not be irreducible. It is, in fact, easy to see that a reduction must always take place at every second step. Observing that  $GF$  only expresses the numerator of the absolute reciprocant  $\frac{1}{\sqrt{t}} \cdot \frac{d}{dx} \left( \frac{F}{t^{\frac{\mu}{2}}} \right)$  in a convenient form, and that  $G^2F$  is equivalent to the numerator of  $\left( \frac{1}{\sqrt{t}} \cdot \frac{d}{dx} \right)^2 \left( \frac{F}{t^{\frac{\mu}{2}}} \right)$ , we have

$$\begin{aligned} \frac{1}{\sqrt{t}} \cdot \frac{d}{dx} \left( \frac{F}{t^{\frac{\mu}{2}}} \right) &= \frac{t \frac{dF}{dx} - \frac{\mu}{2} \cdot aF}{t^{\frac{\mu}{2}+3}}; \\ \text{so that } \left( \frac{1}{\sqrt{t}} \cdot \frac{d}{dx} \right)^2 \left( \frac{F}{t^{\frac{\mu}{2}}} \right) &= \frac{1}{\sqrt{t}} \cdot \frac{d}{dx} \left( \frac{t \frac{dF}{dx} - \frac{\mu}{2} \cdot aF}{t^{\frac{\mu}{2}+3}} \right) \\ &= \frac{t \frac{d}{dx} \left( t \frac{dF}{dx} - \frac{\mu}{2} \cdot aF \right) - \frac{\mu+3}{2} \cdot a \left( t \frac{dF}{dx} - \frac{\mu}{2} \cdot aF \right)}{t^{\frac{\mu}{2}+3}}. \end{aligned}$$

The whole of this fraction is an absolute reciprocant of the same character as  $F$ ; so also is  $\frac{a^2 F}{t^{\frac{\mu}{2}+3}}$  (the product of the *even* absolute reciprocant  $\frac{a^2}{t^3}$  by  $\frac{F}{t^{\frac{\mu}{2}}}$ ). We may therefore reject the term  $\frac{\mu}{2} \cdot \frac{\mu+3}{2} \cdot a^2 F$  from the numerator, and the remaining fraction

$$\frac{\frac{d}{dx} \left( t \frac{dF}{dx} - \frac{\mu}{2} \cdot aF \right) - \frac{\mu+3}{2} \cdot a \frac{dF}{dx}}{t^{\frac{\mu}{2}+2}}$$

will still be an absolute reciprocant of the same character as  $F$ . Its numerator, which is one degree lower than  $G^2F$ , may be written in the form

$$t \frac{d^2 F}{dx^2} - (\mu + \frac{1}{2}) a \frac{dF}{dx} - \frac{\mu}{2} bF.$$

This, it may be noticed, is a reciprocant of the same character as  $F$ , even when  $F$  is non-homogeneous.

Starting with  $a$ , we have

$$Ga = 2bt - 3a^2 \text{ (the Schwarzian),}$$

$$G^2a = G(2bt - 3a^2) = -6a(2bt - 3a^2) + 2t(2ct - 4ab) = 4ct^2 - 20abt + 18a^3.$$

But, for the reason previously given,  $18a^3$  may be removed, so that rejecting this term and dividing out by  $4t$  we obtain the form

$$ct - 5ab,$$

which may be called the Post-Schwarzian.

The next form is obtained by operating on the Post-Schwarzian with  $G$ ; thus, we have to calculate the value of  $G(ct - 5ab)$ , where

$$G = (2bt - 3a^2)\partial_a + (2ct - 4ab)\partial_b + (2dt - 5ac)\partial_c.$$

The working may be arranged as follows:

	$dt^2$	$act$	$b^2t$	$a^2b$	
$t(2dt - 5ac) =$	2	-5	.	.	from $(2dt - 5ac)\partial_c$
$-5a(2ct - 4ab) =$	.	-10	.	20	“ $(2ct - 4ab)\partial_b$
$-5b(2bt - 3a^2) =$	.	.	-10	15	“ $(2bt - 3a^2)\partial_a$
	2	-15	-10	35	

The result should be read thus:

$$2dt^2 - 15act - 10b^2t + 35a^2b.$$

To obtain the next of this series of reciprocants, we have to operate on this with  $G$  and at the same time to take account of the reduction that has to be made at each alternate step. The arrangement of the work is similar to that of the former case.

	$et^3$	$adt^2$	$bct^2$	$a^2ct$	$ab^2t$	$a^3b$	
$2t^2(2et - 6ad) =$	4	-12	.	.	.	.	from $(2et - 6ad)\partial_e$
$-15at(2dt - 5ac) =$	.	-30	.	75	.	.	“ $(2dt - 5ac)\partial_c$
$(35a^2 - 20bt)(2ct - 4ab) =$	.	.	-40	70	80	-140	“ $(2ct - 4ab)\partial_b$
$(70ab - 15ct)(2bt - 3a^2) =$	.	.	-30	45	140	-210	“ $(2bt - 3a^2)\partial_a$
	4	-42	-70	190	220	-350	
$-70a^2(ct - 5ab) =$	.	.	.	-70	.	+350	
	4	-42	-70	120	220	.	

This divides by  $2t$ , giving the reduced value

$$2et^2 - 21adt - 35bct + 60a^2c + 110ab^2.$$

The next obtained by this process will be seen by the following work to be  $4ft^3 - 56act^2 - 112bdt^2 - 70c^2t^2 + 309a^2dt + 995abct + 220b^3t - 660a^3c - 1210a^2b^2$ .

	$ft^3$	$aet^2$	$bdt^2$	$c^2t^2$	$a^2dt$	$abct$	$b^3t$	$a^3c$	$a^2b^2$	
$2t^2(2ft - 7ae) =$	4	-14	.	.	.	.	.	.	.	from $(2ft - 7ae)\partial_e$
$-21at(2et - 6ad) =$	.	-42	.	.	126	.	.	.	.	" $(2et - 6ad)\partial_a$
$(-35bt + 60a^2)(2dt - 5ac) =$	.	.	-70	.	120	175	.	-300	.	" $(2dt - 5ac)\partial_c$
$(-35ct + 220ab)(2ct - 4ab) =$	.	.	.	-70	.	580	.	.	-880	" $(2ct - 4ab)\partial_b$
$(-21dt + 120ac + 110b^2)(2bt - 3a^2) =$	.	.	-42	.	63	240	220	-360	-330	" $(2bt - 3a^2)\partial_a$
	4	-56	-112	-70	309	995	220	-660	-1210	

This cannot be reduced in the same manner as the preceding form, but it must not be supposed that the forms thus obtained are in general irreducible.

Having regard to the circumstance that the forms of the series  $a, Ga, G^2a, \dots$  occur in the numerators of the successive values of  $\left(\frac{1}{\sqrt{t}} \cdot \frac{d}{dx}\right)^n \log t$ , they may be called the successive *educts*, and the reduced forms given above may be called the *reduced educts* and denoted by  $E_1, E_2, E_3, \dots$ . Thus,

$$\begin{aligned}
 E_1 &= a, \\
 E_2 &= 2bt - 3a^2, \\
 E_3 &= ct - 5ab, \\
 E_4 &= 2dt^2 - 15act - 10b^2t + 35a^2b, \\
 E_5 &= 2et^2 - 21adt - 35bct + 60a^2c + 110ab^2, \\
 E_6 &= 4ft^3 - 56aet^2 - 112bdt^2 - 70c^2t^2 + 309a^2dt + 995abct \\
 &\quad + 220b^3t - 660a^3c - 1210a^2b^2.
 \end{aligned}$$

#### LECTURE IV.

We have seen that when  $F$  is a rational integral homogeneous and isobaric reciprocal,  $GF$  is another of the same character. It will now appear that the condition of isobarism is implied in that of homogeneity; for let  $F$  be a rational integral homogeneous reciprocal,  $\mu$  its characteristic and  $i$  its degree in the letters  $t, a, b, c, \dots$ , then, in the identical equation

$$F(t, a, b, c, \dots) = \pm t^\mu F(\tau, \alpha, \beta, \gamma, \dots)$$

both members are homogeneous and of the same degree in the letters  $t, a, b, c, \dots$ ; *i. e.* if  $At^k a^l b^m c^n \dots$  be any term of  $F(t, a, b, c, \dots)$ , its

degree must be the same as that of  $t^{\mu} A \tau^k \alpha^l \beta^m \gamma^n \dots$  when  $\tau, \alpha, \beta, \gamma, \dots$  are expressed in terms of  $t, a, b, c, \dots$ . But

$$\tau = \frac{1}{t}, \quad \alpha = -\frac{a}{t^2}, \quad \beta = -\frac{b}{t^3} + \dots, \quad \gamma = -\frac{c}{t^4} + \dots,$$

and so on. The degrees of  $\tau, \alpha, \beta, \gamma, \dots$  are therefore  $-1, -2, -3, -4, \dots$  respectively. Hence

$$k + l + m + n + \dots = \mu - k - 2l - 3m - 4n - \dots,$$

or

$$\mu = 2k + 3l + 4m + 5n + \dots$$

And by hypothesis  $i = k + l + m + n + \dots$ ,

so that  $\mu - 3i = -k + m + 2n + \dots$

Neither  $\mu$  nor  $i$  is dependent for its value on the selection of a particular term of  $F$ , for all terms of  $F(\tau, \alpha, \beta, \gamma, \dots)$  are multiplied by the same extraneous factor  $\pm t^{\mu}$ , and all terms of  $F(t, a, b, c, \dots)$  are of the same degree  $i$ . Hence  $-k + m + 2n + \dots$  must also be the same for each term of  $F$ ; or, attributing the weights  $-1, 0, 1, 2, \dots$  to the letters  $t, a, b, c, \dots$ , the function  $F$  is isobaric.

Next, suppose  $F$  to be fractional, and let it be the ratio of the two rational integral homogeneous reciprocants  $F_1$  and  $F_2$ . The operation of  $G$  on  $F$  will, in this case also, generate another reciprocant of the same character as  $F$ . For, since  $G$  is linear in the differential operative symbols  $\partial_a, \partial_b, \partial_c, \dots$ , its operation will be precisely analogous to that of differentiation, so that, operating with  $G$  on

$$F = \frac{F_1}{F_2},$$

we have

$$GF = \frac{F_2 GF_1 - F_1 GF_2}{F_2^2}.$$

In order to prove that this is a reciprocant, we have to show that the character and characteristic are the same for both terms of the numerator. But  $GF_1$  is a reciprocant of the same character as  $F_1$ , and  $GF_2$  is one of the same character as  $F_2$ ; thus the two terms of the numerator are of the same character as  $F_1 F_2$ . As regards the characteristic, it should be noticed that  $G$  [*i. e.* the operator  $(2bt - 3a^2)\partial_a + (2ct - 4ab)\partial_b + \dots$ ] increases the degree by unity, but does not alter the weight, so that it increases the characteristic of any rational integral homogeneous reciprocant by 3. Thus the characteristic of each term in the numerator exceeds by 3 that of  $F_1 F_2$ . Hence  $GF$  is a reciprocant,

and, taking account of its denominator as well as its numerator, we see that the operation of  $G$  on a rational homogeneous reciprocant, whether fractional or integral, produces another in which the original character is preserved while the characteristic is increased by three units.

More generally, let  $F_1, F_2, F_3, \dots$  be any rational homogeneous reciprocants whose extraneous factors are  $(-)^{\kappa_1 t^{\mu_1}}, (-)^{\kappa_2 t^{\mu_2}}, (-)^{\kappa_3 t^{\mu_3}}, \dots$  respectively; and suppose  $\Phi$  to consist of a series of terms of the form  $A F_1^{\lambda_1} F_2^{\lambda_2} F_3^{\lambda_3} \dots$ , such that the extraneous factor for each term is  $(-)^{\kappa t^{\mu}}$ . Then  $\Phi$  is a reciprocant, but not necessarily a rational one; for the indices  $\lambda_1, \lambda_2, \lambda_3, \dots$  may be supposed fractional, provided only that they satisfy the conditions  $\kappa_1 \lambda_1 + \kappa_2 \lambda_2 + \kappa_3 \lambda_3 + \dots - \kappa =$  a positive or negative *even* integer, and

$$\mu_1 \lambda_1 + \mu_2 \lambda_2 + \mu_3 \lambda_3 + \dots - \mu = 0.$$

We proceed to show that  $G\Phi$  is also a reciprocant, and that its extraneous factor is  $(-)^{\kappa t^{\mu} + 3}$ . Since

$$G\Phi = \frac{d\Phi}{dF_1} \cdot GF_1 + \frac{d\Phi}{dF_2} \cdot GF_2 + \frac{d\Phi}{dF_3} \cdot GF_3 + \dots,$$

we have to prove not only that each term of this expression is a reciprocant, but also that all of them have the same extraneous factor; otherwise their sum would not be a reciprocant.

Now, in

$$\Phi = \Sigma A F_1^{\lambda_1} F_2^{\lambda_2} F_3^{\lambda_3} \dots,$$

the extraneous factor for each term is by hypothesis  $(-)^{\kappa t^{\mu}}$ , so that the extraneous factor for each term of

$$\frac{d\Phi}{dF_1} = \Sigma A \lambda_1 F_1^{\lambda_1 - 1} F_2^{\lambda_2} F_3^{\lambda_3} \dots$$

is  $(-)^{\kappa - \kappa_1 t^{\mu} - \mu_1}$ , and therefore  $\frac{d\Phi}{dF_1}$  is a reciprocant. Also,  $GF_1$  is a reciprocant

whose extraneous factor is  $(-)^{\kappa_1 t^{\mu_1} + 3}$ . Hence  $\frac{d\Phi}{dF_1} \cdot GF_1$  is a reciprocant having  $(-)^{\kappa t^{\mu} + 3}$  for extraneous factor, and in exactly the same way we see that every other term of  $G\Phi$  is also a reciprocant with the same extraneous factor.

Thus  $G$ , operating on *any* homogeneous reciprocant whose extraneous factor is  $(-)^{\kappa t^{\mu}}$ , generates another whose extraneous factor is  $(-)^{\kappa t^{\mu} + 3}$ .

If, in the generator for mixed reciprocants,

$$G = (2bt - 3a^2)\partial_a + (2ct - 4ab)\partial_b + (2dt - 5ac)\partial_c + \dots,$$

we write

$$a = 1.2.a_0, \quad b = 1.2.3.a_1, \quad c = 1.2.3.4.a_2 \dots,$$

(i. e. if we use the system of modified letters previously mentioned), its expression assumes a more elegant form. Substituting for  $a, b, c, \dots$  their values in terms of the modified letters, we have

$$2bt - 3a^2 = 2.1.2.3a_1t - 3(1.2)^2a_0^2 = 1.2^2.3(a_1t - a_0^2)$$

and

$$\partial_a = \frac{1}{1.2} \cdot \partial_{a_0};$$

so that

$$(2bt - 3a^2)\partial_a = 1.2.3(a_1t - a_0^2)\partial_{a_0}.$$

Again,

$$(2ct - 4ab) = 1.2^2.3.4(a_2t - a_0a_1)$$

and

$$\partial_b = \frac{1}{1.2.3}\partial_{a_1};$$

so that

$$(2ct - 4ab)\partial_b = 1.2.4(a_2t - a_0a_1)\partial_{a_1}.$$

Similarly,

$$(2dt - 5ac)\partial_c = 1.2.5(a_3t - a_0a_2)\partial_{a_2}.$$

Thus the modified generator for mixed reciprocants is

$$1.2.3(a_1t - a_0^2)\partial_{a_0} + 1.2.4(a_2t - a_0a_1)\partial_{a_1} + 1.2.5(a_3t - a_0a_2)\partial_{a_2} + \dots,$$

in which the general term is

$$1.2(n+3)(a_{n+1}t - a_0a_n)\partial_{a_n}.$$

The factor 1.2 may, of course, be rejected, and our modified generator may be written in the simple form

$$3(a_1t - a_0^2)\partial_{a_0} + 4(a_2t - a_0a_1)\partial_{a_1} + 5(a_3t - a_0a_2)\partial_{a_2} + \dots$$

Operating with this on the homogeneous reciprocant  $F(t, a_0, a_1, a_2, \dots)$ , the result will be another homogeneous reciprocant of the same character as  $F$ . When we start with  $a_0$  and make the reductions which, as we have seen, occur at every second step, we find a system of reduced educts corresponding in every particular with those formerly given, but expressed in terms of the modified letters  $a_0, a_1, a_2, \dots$  instead of  $a, b, c, \dots$ . These are as follows:

$$\begin{aligned} & a_0, \\ * & a_1t - a_0^2, \\ & 2a_2t - 5a_0a_1, \\ * & 2a_3t^2 - 6a_0a_2t - 3a_1^2t + 7a_0^2a_1, \\ & 2a_4t^2 - 7a_0a_3t - 7a_1a_2t + 8a_0^2a_2 + 11a_0a_1^2, \\ * & 14a_5t^3 - 56a_0a_4t_2 - 56a_1a_3t^2 - 28a_2^2t^2 + 103a_0^2a_3t + 199a_0a_1a_2t \\ & \quad + 33a_1^3t - 88a_0^3a_2 - 121a_0^2a_1^2. \\ & \dots \end{aligned}$$

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\* It will be observed that in the unreduced forms, marked with an asterisk, the sum of the numerical coefficients is zero. This is a direct consequence, as may be easily seen, of the form of the modified generator, in which the sum of the numerical coefficients in each term is also zero.

It will be found on trial that these modified educts are obtained with greater ease and with less liability to error by a direct application of the generator

$$3(a_1t - a_0^2)\partial_{a_0} + 4(a_2t - a_0a_1)\partial_{a_1} + 5(a_3t - a_0a_2)\partial_{a_2} + \dots,$$

than by making the substitution of  $1.2.a_0$ ,  $1.2.3.a_1$ ,  $1.2.3.4.a_2$ ,  $\dots$  for  $a$ ,  $b$ ,  $c$ ,  $\dots$  in the system of educts already given. For this reason the working by the former method is here performed, instead of being merely indicated.

From  $a_0$  we obtain immediately

$$a_1t - a_0^2.$$

Operating on this with the generator, there results

$$4t(a_2t - a_0a_1) - 6a_0(a_1t - a_0^2) = 4a_2t^2 - 10a_0a_1t + 6a_0^3.$$

This, when reduced by removing its last term and dividing the others by  $2t$ , gives

$$2a_2t - 5a_0a_1.$$

The next form is found from this by a simple operation, without subsequent reduction, and is therefore

$$10t(a_3t - a_0a_2) - 20a_0(a_2t - a_0a_1) - 15a_1(a_1t - a_0^2).$$

Or, collecting the terms and rejecting the numerical factor 5,

$$2a_3t^2 - 6a_0a_2t - 3a_1^2t + 7a_0^2a_1.$$

The operation of the generator on this gives

$$12t^2(a_4t - a_0a_3) - 30a_0t(a_3t - a_0a_2) + 4(7a_0^2 - 6a_1t)(a_2t - a_0a_1) + 3(14a_0a_1 - 6a_2t)(a_1t - a_0^2).$$

The collection of terms and subsequent reduction is shown below :

	$a_4t^3$	$a_0a_3t^2$	$a_1a_2t^2$	$a_0^2a_2t$	$a_0a_1^2t$	$a_0^3a_1$
	12	-12	.	.	.	.
	.	-30	.	30	.	.
	.	.	-24	28	24	-28
	.	.	-18	18	42	-42
	12	-42	-42	76	66	-70
$-14a_0^2(2a_2^2t - 5a_0a_1) =$	.	.	.	-28	.	+70
	12	-42	-42	48	66	

Removing the factor  $6t$ , the reduced form is

$$2a_4t^2 - 7a_0a_3t - 7a_1a_2t + 8a_0^2a_2 + 11a_0a_1^2.$$

Operating on this with the generator, we have

$$\begin{aligned} & 14t^2(a_5t - a_0a_4) - 42a_0t(a_4t - a_0a_3) + 5(8a_0^2 - 7a_1t)(a_3t - a_0a_2) \\ & + 4(22a_0a_1 - 7a_2t)(a_2t - a_0a_1) + 3(11a_1^2 + 16a_0a_2 - 7a_3t)(a_1t - a_0^2) \\ & = 14a_5t^3 - 56a_0a_4t^2 - 56a_1a_3t^2 - 28a_2^2t^2 + 103a_0^2a_3t + 199a_0a_1a_2t \\ & + 33a_1^3t - 88a_0^3a_2 - 121a_0^2a_1^2, \end{aligned}$$

which cannot be reduced in the same manner as the preceding form.

To obtain a generator for passing from pure to pure reciprocants a process is employed similar to that which gave the generator for mixed reciprocants which we have just been using. I state the results before giving the proof, and then proceed to speak of generators in the theory of Invariants. The generator for pure reciprocants is

$$(3ac - 4b^2)\partial_b + (3ad - 5bc)\partial_c + (3ae - 6bd)\partial_d + \dots;$$

or, expressed in terms of the modified letters,

$$4(a_0a_2 - a_1^2)\partial_{a_1} + 5(a_0a_3 - a_1a_2)\partial_{a_2} + 6(a_0a_4 - a_1a_3)\partial_{a_3} + \dots$$

By operating with this on any pure reciprocant  $R$ , we generate another pure reciprocant of opposite character to that of  $R$ .

The connection between the two theories of Reciprocants and Invariants is so close, and these brother-and-sister theories throw so much light upon each other, that I began to inquire whether, in the latter, there did not exist a theory of Generators parallel to that of the former.

Fortunately, Mr. Hammond was able to recall a correspondence in which Prof. Cayley had given such a theory, which he regarded, and justly, as an important invention. Its substance has been subsequently incorporated in the *Quarterly Journal* (Vol. XX, p. 212). It offers itself spontaneously in the Reciprocative Theory; in the Invariantive one it calls for a distinct act of invention. Prof. Cayley has discovered two generators similar in form with those for reciprocants, and one of them strikingly so; in a letter to me he calls these  $P$  and  $Q$ . As given by him,

$$\begin{aligned} P &= ab\partial_a + ac\partial_b + ad\partial_c + \dots - ib, \\ Q &= \quad \quad ac\partial_b + 2ad\partial_c + \dots - 2wb, \end{aligned}$$

where  $i$  is the degree and  $w$  the weight, the weights of  $a, b, c, d, \dots$  being taken to be  $0, 1, 2, 3, \dots$  (I supply the  $a$  which Cayley turns into unity.) As an example he takes the "Invariant"  $a^2d - 3abc + 2b^3 = I$ , suppose. We have then

$$\begin{aligned} PI &= (ab\partial_a + ac\partial_b + ad\partial_c + ae\partial_d - 3b)I \\ &= ab(2ad - 3bc) + ac(-3ac + 6b^2) - 3a^2bd + a^3e - 3b(a^2d - 3abc + 2b^3) \\ &= a^3e - 4a^2bd - 3a^2c^2 + 12ab^2c - 6b^4 \\ &= a^2(ac - 4bd + 3c^2) - 6(ac - b^2)^2, \end{aligned}$$

$$\begin{aligned}
\text{and} \quad QI &= (ac\partial_b + 2ad\partial_c + 3ae\partial_d - 6b)I \\
&= ac(-3ac + 6b^2) - 6a^2bd + 3a^3e - 6b(a^2d - 3abc + 2b^3) \\
&= 3a^3e - 12a^2bd - 3a^2c^2 + 24ab^2c - 12b^4 \\
&= 3a^2(ae - 4bd + 3c^2) - 12(ac - b^2)^2.
\end{aligned}$$

$P$  and  $Q$  may be transformed by means of Euler's equation and the similar one for isobaric functions, which enable us to write

$$i = a\partial_a + b\partial_b + c\partial_c + d\partial_d + \dots$$

$$\text{and} \quad w = b\partial_b + 2c\partial_c + 3d\partial_d + \dots;$$

$$\begin{aligned}
P \text{ thus becomes} \quad & ab\partial_a + ac\partial_b + ad\partial_c + ae\partial_d + \dots \\
& - ab\partial_a - b^2\partial_b - bc\partial_c - bd\partial_d - \dots \\
& = (ac - b^2)\partial_b + (ad - bc)\partial_c + (ae - bd)\partial_d + \dots,
\end{aligned}$$

the same in *form* as either of our generators, except that the arithmetical coefficients are all made units;  $a, b, c, \dots$  taking the place of the  $t, a, b, \dots$  of the generator for mixed reciprocants.

In like manner,  $Q$  becomes

$$(ac - 2b^2)\partial_b + 2(ad - 2bc)\partial_c + 3(ae - 2bd)\partial_d + \dots,$$

where the arithmetical series  $1, 2, 3, \dots$  takes the place of  $3, 4, 5, \dots$  or of  $4, 5, 6, \dots$  in the two Reciprocant Generators.

The effect of  $P$  and of  $Q$  is obviously to raise the degree and the weight of the operand  $I$  each by one unit. But if we take  $R = \frac{1}{a}(2wP - iQ)$ , the terms in Cayley's original formulae containing  $b$  cancel, so that  $2wP - iQ$  divides out by  $a$  and the weight is raised one unit without the degree being affected. This is mentioned in the *Quarterly Journal* (*loc. cit.*); but it may also be remarked that when  $I$  is a *satisfied invariant*, it is annihilated by the operation of  $R$ ; when the *invariant* is *unsatisfied*, each of the three operators  $P, Q$  and  $R$  increases its extent by an unit, *i. e.* introduces an additional letter. For let  $j$  denote the extent, then, writing  $a_0, a_1, a_2, \dots, a_j$  for  $a, b, c, \dots$ , we have

$$P = a_0a_1\partial_{a_0} + a_0a_2\partial_{a_1} + \dots + a_0a_{j+1}\partial_{a_j} - ia_1,$$

$$Q = a_0a_2\partial_{a_1} + 2a_0a_3\partial_{a_2} + \dots + ja_0a_{j+1}\partial_{a_j} - 2wa_1;$$

whence we find

$$R = \frac{1}{a_0}(2wP - iQ)$$

$$= 2wa_1\partial_{a_0} + (2w - i)a_2\partial_{a_1} + \dots + (2w - ij + i)a_j\partial_{a_{j-1}} + (2w - ij)a_{j+1}\partial_{a_j}.$$

But for a *satisfied invariant*  $2w = ij$ ;

and substituting this value for  $2w$  in the above expression for  $R$ , it becomes

$$i\{ja_1\partial_{a_0} + (j-1)a_2\partial_{a_1} + \dots + a_j\partial_{a_{j-1}}\},$$

which, as is well known, annihilates any satisfied invariant.

## LECTURE V.

It will be desirable to fill up some of the previous investigations by discussing some points in them that have not yet received our consideration.

There may be some to whom it may appear tedious to watch the complete exposition of the algebraical part of the Theory and who are impatient to rush on to its applications. But it is my duty to consider what may be expected to be most useful to the great majority of the class, and for that purpose to make the ground sure under our feet as I proceed. To the greater number it will, I think, be of advantage to have their memories refreshed on the kindred subject of invariants, and probably made acquainted with some important points of that theory which are new to them.

I confess that, to myself, the contemplation of this relationship—the spectacle of a new continent rising from the waters, resembling yet different from the old, familiar one—is a principal source of interest arising out of the new theory. I do not regard Mathematics as a science purely of calculation, but one of ideas, and as the embodiment of a Philosophy. An eminent colleague of mine, in a public lecture in this University, magnifying the importance of classical over mathematical studies, referred to a great mathematician as one who might possibly know every foot of distance between the earth and the moon; and when I was a member, at Woolwich, of the Government Committee of Inventions, one of my colleagues, appealing to me to answer some question as to the number of cubic inches in a pipe, expressed his surprise that I was not prepared with an immediate answer, and said he had supposed that I had all the tables of weights and measures at my fingers' ends.

I hope that in any class which I may have the pleasure of conducting in this University, other ideas will prevail as to the true scope of mathematical science as a branch of liberal learning; and it will be my endeavor to regulate the pace in a manner which seems to me most conducive to real progress in the order of ideas and philosophical contemplation, thus bringing our noble science into harmony and in a line with the prevailing tone and studies of this University. So long as we are content to be regarded as mere calculators (Faraday, at the end of his experimental lectures, was accustomed to say—I have myself heard him do so—"We will now leave that to the calculators"), we shall be the Pariahs of the University, living here on sufferance, instead of

being regarded, as is our right and privilege, as the real leaders and pioneers of thought in it.

That Cayley's two operators, which have been called  $P$  and  $Q$ , are in fact generators, may be proved as follows:\*

Let  $\Omega = a\partial_b + 2b\partial_c + 3c\partial_d + 4d\partial_e + \dots$   
 and  $\Theta = a(\lambda b\partial_a + \mu c\partial_b + \nu d\partial_c + \dots) - \kappa b$ ,  
 where  $\kappa, \lambda, \mu, \nu, \dots$  are numbers.

When  $\kappa$  is the degree of the operand, and  $\lambda = \mu = \nu = \dots = 1$ , the operator  $\Theta$  is identical with  $P$ ; but  $\Theta$  is identical with  $Q$  when  $\kappa$  is twice the weight of the operand and  $\lambda = 0, \mu = 1, \nu = 2, \dots$ .

If now we use  $*$  to signify the act of pure differential operation, it is obvious that

$$\Omega\Theta = (\Omega \times \Theta) + (\Omega * \Theta),$$

$$\Theta\Omega = (\Omega \times \Theta) + (\Theta * \Omega),$$

so that  $\Omega\Theta - \Theta\Omega = (\Omega * \Theta) - (\Theta * \Omega)$ .

But since  $\Omega a = 0, \Omega b = a, \Omega c = 2b, \dots$ ,

we have  $\Omega * \Theta = a(\lambda a\partial_a + 2\mu b\partial_b + 3\nu c\partial_c + \dots - \kappa)$

and  $\Theta * \Omega = a(\lambda b\partial_b + 2\mu c\partial_c + 3\nu d\partial_d + \dots)$ .

Hence  $\Omega\Theta - \Theta\Omega = a\{\lambda a\partial_a + (2\mu - \lambda)b\partial_b + (3\nu - 2\mu)c\partial_c + \dots - \kappa\}$ ;

or if the operand  $I$  be any *invariant* (satisfied or unsatisfied), we have  $\Omega I = 0$ , and therefore  $\Theta\Omega I = 0$ ; so that we find

$$\Omega\Theta I = a\{\lambda a\partial_a + (2\mu - \lambda)b\partial_b + (3\nu - 2\mu)c\partial_c + \dots - \kappa\}I.$$

If in this we write  $\lambda = \mu = \nu = \dots = 1$ , and  $\kappa = i$ , where  $i$  is the degree of the operand,  $\Theta$  becomes  $P$  and we have

$$\Omega P I = a(a\partial_a + b\partial_b + c\partial_c + \dots - i)I.$$

But, by Euler's theorem, the right-hand side of this vanishes, and therefore

$$\Omega P I = 0.$$

Similarly, by means of the corresponding theorem for isobaric functions, we may prove that

$$\Omega Q I = 0.$$

For if, in the general formula, we write  $\lambda = 0, \mu = 1, \nu = 2, \dots$  and  $\kappa = 2w$ , where  $w$  is the weight of the operand, we find

$$\Omega Q I = a(2b\partial_b + 4c\partial_c + 6d\partial_d + \dots - 2w)I = 0.$$

Thus, when  $\Theta$  stands either for  $P$  or for  $Q$ , it is either an annihilator or a

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\* In the *Quarterly Journal* (Vol. XX, p. 212) Prof. Cayley only considers a special example, and has not given the proof of the general theorem.

generator (*i. e.*  $\Theta I$  is either identically zero or else an invariant). But if  $l$  be the most advanced, or say the *radical letter* of  $I$ , no term of  $m\partial_l I$  can cancel with any other term of  $\Theta I$ ; and since, for this reason,  $\Theta I$  cannot vanish identically, it must be an invariant, and the operators  $P$  and  $Q$  must be generators.

The generators previously given for reciprocants also possess this property of introducing a fresh radical letter at each step. The radical letter, on its first introduction, enters in the first degree only, and in the case of the educts of  $\log t$ , whose values have been calculated, its multiplier is seen to be a power of  $t$ . The form of the generator for mixed reciprocants

$$3(a_1 t - a_0^2)\partial_{a_0} + 4(a_2 t - a_0 a_1)\partial_{a_1} + \dots + (n+3)(a_{n+1} t - a_0 a_n)\partial_{a_n}$$

shows this, or it may be seen by considering the successive values of

$$\left(\frac{1}{\sqrt{t}} \cdot \frac{d}{dx}\right)^i \log t.$$

For let  $\frac{F(t, a_0, a_1, a_2, \dots)}{t^i}$  denote this expression, and let its radical letter be  $a_n$ ; then, on differentiating again with respect to  $x$ , the new letter introduced arises solely from a term in the numerator

$$\frac{d}{da_n} F(t, a_0, a_1, a_2, \dots, a_n) \cdot \frac{da_n}{dx}.$$

But  $a_n = \frac{d^ny}{dx^n} \div 2.3 \dots n+2$ ; so that  $\frac{da_n}{dx} = (n+3)a_{n+1}$ .

Hence, if when  $a_n$  is the radical letter, it occurs in the first degree only and multiplied by a power of  $t$ , it follows that, since  $\frac{dF}{da_n}$  will be a power of  $t$ , the derived expression which contains the radical letter  $a_{n+1}$  will contain it in the first degree only and multiplied by a power of  $t$ . And since this is true for the case  $i=1$ , when  $\frac{1}{\sqrt{t}} \cdot \frac{d}{dx} \log t = \frac{a_0}{t^{\frac{3}{2}}}$ , it is true universally.

Observe that for  $i=1, 2, 3, \dots$  the radical letter is  $a_0, a_1, a_2, \dots$  respectively.

It will be remembered that  $\left(\frac{1}{\sqrt{t}} \cdot \frac{d}{dx}\right)^i \log t$  is an absolute reciprocant. It may be called the  $i^{\text{th}}$  absolute educt, to distinguish it from the rational integral educts  $E_1, E_2, E_3, \dots$  whose values have already been calculated.

Let  $R(t, a_0, a_1, a_2, \dots, a_n)$  be any homogeneous rational integral reciprocant, and let the educts be  $A_0, A_1, A_2, \dots, A_n$ ; then obviously

$$\begin{array}{cccccccc} a_n & \text{may be expressed rationally in terms of } A_n \text{ and } a_{n-1}, a_{n-2}, \dots, a_0, t, \\ a_{n-1} & \text{“ “ “ “ “ } A_{n-1} \text{ and } a_{n-2}, \dots, a_0, t, \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_1 & \text{“ “ “ “ “ } A_1, a_0 \text{ and } t, \\ a_0 & \text{“ “ “ “ “ } A_0 \text{ and } t, \end{array}$$

where observe that the denominators in these expressions are all powers of  $t$ . Hence, by successive substitutions,  $R(t, a_0, a_1, \dots, a_n)$  may be expressed rationally in terms of  $A_n, \dots, A_1, A_0$ , and  $t$ . Thus any rational integral homogeneous reciprocant is a rational function of educts, and is of the form  $\frac{E}{t^0}$ , where  $E$  is a rational *integral* function of the educts.

Does not this prove too much, it may be asked, viz.: that any function  $F$  of the letters is a rational function of the educts, which are themselves reciprocants, and will therefore be a reciprocant? But this is not so; for observe that although  $F$  will be expressed as a sum of products of educts, such products will not in general be all of the same character, and their linear combination will be an illicit one, such as is seen in the illicit combination of  $a_0^2$  with the Schwarzian  $(a_1t - a_0^2)$ .

We have seen that by differentiating an absolute reciprocant, or by the use of a generator, we obtain a fresh reciprocant. But there are other methods of finding reciprocants; as, for example, if the transform of  $\phi(t, a, b, c, \dots)$  is  $\psi(\tau, \alpha, \beta, \gamma, \dots)$ , i. e. if

$$\phi(t, a, b, c, \dots) = \psi(\tau, \alpha, \beta, \gamma, \dots),$$

then

$$\psi(t, a, b, c, \dots) = \phi(\tau, \alpha, \beta, \gamma, \dots).$$

Whence, by multiplication,

$$\phi(t, a, b, c, \dots) \psi(t, a, b, c, \dots) = \phi(\tau, \alpha, \beta, \gamma, \dots) \psi(\tau, \alpha, \beta, \gamma, \dots).$$

Thus  $\phi.\psi$  is a reciprocant, and, moreover, an absolute one of even character, although neither  $\phi$ , which is a perfectly arbitrary function, nor  $\psi$ , its transform, is a reciprocant.

Herein a mixed reciprocant differs from an invariant, which cannot be resolved into non-invariantive factors. It is worth while to give a proof of this proposition; but first I prove its converse, that if  $p, q, r, \dots$  are all invariants, their product must be so too. This is an immediate consequence of the well-known theorem that

$$\Omega I = 0$$

is the necessary and sufficient condition that  $I$  may be an invariant where, as usual,  $\Omega$  is the operator

$$a\partial_b + 2b\partial_c + 3c\partial_d + \dots,$$

and the word invariant has been used in the same extended sense as formerly.

For 
$$\Omega(pqrs \dots) = \left( \frac{\Omega p}{p} + \frac{\Omega q}{q} + \frac{\Omega r}{r} + \dots \right) pqrs \dots$$

But since  $p, q, r, \dots$  are all invariants, we have  $\Omega p = 0, \Omega q = 0, \Omega r = 0, \dots$ , and therefore

$$\Omega(pqrs \dots) = 0.$$

Next, suppose that

$$I = P_1 Q_1,$$

where  $I$  is but  $Q_1$  is not an invariant.

To meet the case in which  $P_1$  and  $Q_1$  are not prime to one another,  $Q_1$ , if resolved into its factors, must contain one  $Q^i$  where  $Q$  is not an invariant.

Suppose that  $P_1$  contains  $Q^j$ , and let  $i + j = k$ ; then we may write

$$I = P Q^k,$$

where  $P$  is prime to  $Q$ . But since  $I$  is an invariant by hypothesis,

$$\Omega I = 0,$$

and therefore,

$$Q^k \Omega P + k P Q^{k-1} \Omega Q = 0;$$

or,

$$\frac{Q}{P} = -k \frac{\Omega Q}{\Omega P}.$$

Now  $P$  is prime to  $Q$ , so that the fraction  $\frac{Q}{P}$  is in its lowest terms; therefore

$\Omega Q$  contains  $Q$ ; but this is impossible, for the weight of  $\Omega Q$  is less than that of  $Q$ . Hence  $I$  cannot contain any non-invariantive factor  $Q_1$ .

All this will be equally true for a general function  $J$  annihilated by any operator  $\Omega$  which is *linear* in the differential operators  $\partial_a, \partial_b, \partial_c, \dots$  no matter what its degree in the letters  $a, b, c, \dots$  themselves; *i. e.* we shall still have

$$J = P Q^k$$

and

$$\frac{Q}{P} = -k \frac{\Omega Q}{\Omega P},$$

where  $P$  and  $Q$  are prime to each other, and, as before,  $\Omega Q$  will contain  $Q$  as a factor. But if  $\Omega$  is an operator which diminishes either the degree or the weight,  $\Omega Q$  is either of lower degree or of lower weight than  $Q$ , and so cannot contain it as a factor. Hence  $J$  cannot contain a factor  $Q$  not subject to annihilation by  $\Omega$ .

If, however,  $\Omega$  does not diminish either the degree or the weight, it may be objected that  $\Omega Q$  might conceivably contain the factor  $Q$ ; and were it so, there would be nothing to show the impossibility, in this case, of a function  $J$  subject to annihilation by  $\Omega$  containing a factor  $Q$ , which is not so. But *quaere*: Is it possible, when  $J$  is a general homogeneous and isobaric function of  $a, b, c, \dots$ , for  $\Omega J$  to contain  $J$  and at the same time the quotient to be other than a

number? \* *Valde dubitor.* But I reserve the point. Setting aside this doubtful case, and considering only such *linear* partial differential operators as *diminish* either the degree or the weight of the operand, we see that there cannot exist any universal operator of this kind whose effect in annihilating a form is the necessary and sufficient condition of that form being a reciprocant. But this does not preclude the possibility of the existence of such annihilators for special classes of reciprocants, and in fact (as we have already stated and shall hereafter prove) Pure Reciprocants are definable by means of the Partial Differential Annihilator

$$V = 4 \cdot \frac{a_0^2}{2} \partial_{a_1} + 5a_0a_1\partial_{a_2} + 6\left(a_0a_2 + \frac{a_1^2}{2}\right)\partial_{a_3} + \dots,$$

which is *linear* in the differential operators, and *diminishes* the weight.

The generator for mixed reciprocants, which we have called  $G$ , will not assist us in obtaining pure reciprocants, but generates a mixed reciprocant in every case, even when the one we start with is pure. Thus, starting with the pure reciprocant  $R$ , our formula

$GR = \{3(a_1t - a_0^2)\partial_{a_0} + 4(a_2t - a_0a_1)\partial_{a_1} + 5(a_3t - a_0a_2)\partial_{a_2} + \dots\} R$   
may be written thus

$$GR = t(3a_1\partial_{a_0} + 4a_2\partial_{a_1} + 5a_3\partial_{a_2} + \dots)R \\ - a_0(3a_0\partial_{a_0} + 4a_1\partial_{a_1} + 5a_2\partial_{a_2} + \dots)R.$$

Here  $R$  being *pure*, i. e. a function of  $a_0, a_1, a_2, \dots$  (without  $t$ ), we see that

$$(3a_0\partial_{a_0} + 4a_1\partial_{a_1} + 5a_2\partial_{a_2} + 6a_3\partial_{a_3} + \dots)R \\ = 3(a_0\partial_{a_0} + a_1\partial_{a_1} + a_2\partial_{a_2} + \dots)R \\ + (a_1\partial_{a_1} + 2a_2\partial_{a_2} + 3a_3\partial_{a_3} + \dots)R \\ = (3i + w)R,$$

where  $i$  is the degree and  $w$  the weight of  $R$ . Hence

$$GR = t(3a_1\partial_{a_0} + 4a_2\partial_{a_1} + 5a_3\partial_{a_2} + \dots)R - (3i + w)a_0R,$$

where it should be noticed that  $a_0R$  is of opposite character to  $R$  (for  $a_0$  is of odd character), while  $GR$  has been proved to be of the same character as  $R$ . Thus we cannot infer that  $t(3a_1\partial_{a_0} + 4a_2\partial_{a_1} + 5a_3\partial_{a_2} + \dots)R$  is a reciprocant. The mixed reciprocant  $GR$  cannot therefore be resolved into the sum of two terms, one of which is a pure reciprocant and the other a pure reciprocant multiplied by  $t$ .

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\* If  $\Omega = pa\partial_a + qb\partial_b + rc\partial_c + \dots$ , where  $p, q, r, \dots$  are in Arithmetical Progression,  $\frac{\Omega J}{J}$  is a number; but then  $\Omega$  could not be an annihilator.

## LECTURE VI.

Before proceeding to prove that, as was stated in anticipation in Lecture IV, the operator

$$(3ac - 4b^2) \partial_b + (3ad - 5bc) \partial_c + (3ae - 6bd) \partial_d + \dots,$$

or, when the modified letters are used,

$$4(a_0a_2 - a_1^2) \partial_{a_1} + 5(a_0a_3 - a_1a_2) \partial_{a_2} + 6(a_0a_4 - a_1a_3) \partial_{a_3} + \dots,$$

will serve to generate a pure reciprocant from a pure one, it may be useful to briefly recapitulate what has been said concerning the character and characteristic of reciprocants. It will be remembered that the extraneous factor of any rational integral reciprocant is of the form  $(-)^{\kappa} t^{\mu}$ , that the character is determined by the parity (oddness or evenness) of  $\kappa$ , and that  $\mu$  is what has been called the characteristic.

For homogeneous reciprocants it has been proved that  $\mu = 3i + w$ , where  $i$  is the degree of the reciprocant and  $w$  its weight, the weights of the letters  $t, a, b, c, \dots$  being taken to be  $-1, 0, 1, 2, \dots$  respectively. The character is odd or even according as the number of letters other than  $t$  in the principal term or terms is odd or even. By a principal term is to be understood one in which  $t$  is contained the greatest number of times. So that, in other words, the character is governed by the parity of the smallest number of non- $t$  letters that can be found in any term. For pure reciprocants, there being no  $t$  in any term, the character is determined by the parity of the number of letters in any one term.

Let  $R$  be any pure reciprocant, and suppose its characteristic to be  $\mu$ ; then  $\frac{R}{t^{\frac{\mu}{2}}}$  is an absolute reciprocant. If, however, we differentiate this with respect to  $x$ , and thus obtain another reciprocant, the resulting form will not be pure, for its numerator will be identical with the form obtained by the direct operation on  $R$  of the generator for mixed reciprocants, and its denominator will be a power of  $t$ . But, remembering that  $\frac{a}{t^{\frac{3}{2}}}$ , and therefore  $\frac{a^{\frac{\mu}{3}}}{t^{\frac{\mu}{2}}}$ , is an absolute reciprocant,

we see that  $\frac{R}{a^{\frac{\mu}{3}}}$ , which is the quotient of the two absolute reciprocants  $\frac{R}{t^{\frac{\mu}{2}}}$  and  $\frac{a^{\frac{\mu}{3}}}{t^{\frac{\mu}{2}}}$ ,

is so also. Hence  $\frac{d}{dx} \left( \frac{R}{a^{\frac{\mu}{3}}} \right)$  is a reciprocant, and, since it no longer contains  $t$ , a pure one. Now,

$$\frac{d}{dx} \left( \frac{R}{a^{\frac{\mu}{3}}} \right) = \frac{a \frac{dR}{dx} - \frac{\mu}{3} \cdot bR}{a^{\frac{\mu}{3} + 1}}$$

remains a reciprocant when multiplied by any power of the reciprocant  $a$ . Hence the numerator of this expression, or

$$\left(3a \frac{d}{dx} - \mu b\right) R,$$

is a reciprocant. The general value of  $\frac{d}{dx}$  has been seen to be

$$a\partial_t + b\partial_a + c\partial_b + d\partial_c + \dots,$$

but, since  $R$  is supposed to be *pure*,  $\partial_t R = 0$ .

We may therefore, in  $3a \frac{d}{dx} - \mu b$ , replace  $\frac{d}{dx}$  by

$$b\partial_a + c\partial_b + d\partial_c + e\partial_d + \dots.$$

Now, remembering that  $\mu = 3i + w$ , and that by Euler's theorem and the similar one for isobaric functions

$$i = a\partial_a + b\partial_b + c\partial_c + d\partial_d + \dots$$

$$\text{and} \quad w = b\partial_b + 2c\partial_c + 3d\partial_d + \dots,$$

we see that  $\mu$  is equivalent to

$$3a\partial_a + 4b\partial_b + 5c\partial_c + 6d\partial_d + \dots.$$

$$\begin{aligned} \text{Hence,} \quad 3a \frac{d}{dx} - \mu b &= 3a(b\partial_a + c\partial_b + d\partial_c + e\partial_d + \dots) \\ &\quad - b(3a\partial_a + 4b\partial_b + 5c\partial_c + 6d\partial_d + \dots) \\ &= (3ac - 4b^2)\partial_b + (3ad - 5bc)\partial_c + (3ae - 6bd)\partial_d + \dots \end{aligned}$$

Thus, if  $R$  be any *pure* reciprocant,

$$\{(3ac - 4b^2)\partial_b + (3ad - 5bc)\partial_c + (3ae - 6bd)\partial_d + \dots\} R$$

is also a pure reciprocant. If the type of  $R$  be  $w; i, j$ , that of the form derived from it will clearly be  $w + 1; i + 1, j + 1$ . Its character (which, for pure reciprocants, depends solely on the degree) will therefore be opposite to that of  $R$ , and its characteristic will be  $\mu + 4$ , that of  $R$  being  $\mu$ .

Beginning with the form  $3ac - 5b^2$ , which was given as an example in Lecture II, a series of pure "educts" may be obtained by the repeated use of the above generator; and it will be noticed that the successive educts thus formed are alternately of even and odd character, whereas those previously given, viz.  $a, 2bt - 3a^2, \dots$ , were all negative. A reduction similar to that which formerly took place when the generator for mixed reciprocants was used, may be effected at each second step in the present case. For, since the characteristic of  $\left(3a \frac{d}{dx} - \mu b\right) R$  is  $\mu + 4$ , the next operation will give

$$\left(3a \frac{d}{dx} - (\mu + 4)b\right) \left(3a \frac{d}{dx} - \mu b\right) R.$$

Performing the indicated differentiations, this becomes

$$\begin{aligned} & 3a \frac{d}{dx} \left( 3a \frac{dR}{dx} - \mu b R \right) - 3(\mu + 4)ab \frac{dR}{dx} + \mu(\mu + 4)b^2 R \\ &= 9a^2 \frac{d^2 R}{dx^2} + 9ab \frac{dR}{dx} - 3\mu ab \frac{dR}{dx} - 3\mu ac R - 3(\mu + 4)ab \frac{dR}{dx} + \mu(\mu + 4)b^2 R \\ &= 9a^2 \frac{d^2 R}{dx^2} - 3(2\mu + 1)ab \frac{dR}{dx} - 3\mu ac R + \mu(\mu + 4)b^2 R. \end{aligned}$$

Adding  $\mu(\mu + 4)(3ac - 5b^2)R$  to 5 times the above expression, we obtain

$$45a^2 \frac{d^2 R}{dx^2} - 15(2\mu + 1)ab \frac{dR}{dx} + 3\mu(\mu - 1)acR,$$

which, when divided by  $3a$ , gives the pure reciprocant

$$15a \frac{d^2 R}{dx^2} - 5(2\mu + 1)b \frac{dR}{dx} + \mu(\mu - 1)cR.$$

This form is one degree lower than the second educt from  $R$ , the depression of degree being due to the removal of a factor  $a$  by division.

When the modified letters  $a_0, a_1, a_2, a_3, \dots$  are used, the generator

$$(3ac - 4b^2)\partial_b + (3ad - 5bc)\partial_c + (3ae - 6bd)\partial_d + \dots \quad (1)$$

is easily transformed by writing in it

$$a = 2a_0, \quad b = 2.3.a_1, \quad c = 2.3.4.a_2, \quad d = 2.3.4.5.a_3 \dots,$$

and consequently

$$\partial_b = \frac{\partial_{a_1}}{2.3}, \quad \partial_c = \frac{\partial_{a_2}}{2.3.4}, \quad \partial_d = \frac{\partial_{a_3}}{2.3.4.5} \dots,$$

when it becomes

$$\frac{2^2.3^2.4}{2.3}(a_0a_2 - a_1^2)\partial_{a_1} + \frac{2^2.3^2.4.5}{2.3.4}(a_0a_3 - a_1a_2)\partial_{a_2} + \frac{2^2.3^2.4.5.6}{2.3.4.5}(a_0a_4 - a_1a_3)\partial_{a_3} + \dots$$

Dividing each term of this by 2.3, and writing the numerical coefficients in their simplest form, we have

$$4(a_0a_2 - a_1^2)\partial_{a_1} + 5(a_0a_3 - a_1a_2)\partial_{a_2} + 6(a_0a_4 - a_1a_3)\partial_{a_3} + \dots, \quad (2)$$

which is the modified generator previously mentioned.

The generators formerly used in the theory of mixed reciprocants were

$$(2tb - 3a^2)\partial_a + (2tc - 4ab)\partial_b + (2td - 5ac)\partial_c + \dots \quad (3)$$

$$\text{and} \quad 3(ta_1 - a_0^2)\partial_{a_0} + 4(ta_2 - a_0a_1)\partial_{a_1} + 5(ta_3 - a_0a_2)\partial_{a_2} + \dots \quad (4)$$

The memory will be assisted in retaining these formulae if we observe that (1) is obtainable from (3), or (2) from (4), by increasing at the same time each numerical coefficient and the weight of each letter by unity.

It will, I think, be instructive to see how the form  $3ac - 5b^2$  was found originally by combining mixed reciprocants. The degree alone of a pure reciprocant suffices, as we have seen, to determine its character; but when we are dealing with mixed reciprocants their character does not depend either on the degree or the weight, so that we require a notation to discriminate between forms of the same degree-weight, but of opposite character. In what follows, (+) placed before any form signifies that it is a reciprocant of *even* character, while (−) signifies that its character is odd.

I have previously given the three *odd* reciprocants

$$(-) \quad a, \quad (A)$$

$$(-) \quad 2bt - 3a^2, \quad (B)$$

$$(-) \quad ct - 5ab. \quad (C)$$

From these we obtain *even* reciprocants; thus the product of (A) and (C) is

$$(+)\quad act - 5a^2b, \quad (D)$$

and the square of (B) is  $(+)\quad 4b^2t^2 - 12a^2bt + 9a^4$ .

After subtracting the *even* reciprocant  $9a^4$  from this, we may remove the factor  $4t$  from the remainder without thereby affecting its character. These reductions give

$$(+)\quad b^2t - 3a^2b,$$

which may be combined with the *even* reciprocant (D) in such a manner that the combination contains a factor  $t$ . In fact,

$$3(act - 5a^2b) - 5(b^2t - 3a^2b) = (3ac - 5b^2)t,$$

so that a *legitimate* combination of mixed reciprocants can be made to give the pure one

$$3ac - 5b^2.$$

Similarly we might find the known form

$$9a^2d - 45abc + 40b^3,$$

which equated to zero expresses Sextactic Contact at a point  $x, y$ . But it is more readily obtained by operating with the generator on  $3ac - 5b^2$ ; thus,

$$\begin{aligned} \{ (3ac - 4b^2)\partial_b + (3ad - 5bc)\partial_c \} (3ac - 5b^2) &= -10b(3ac - 4b^2) + 3a(3ad - 5bc) \\ &= 9a^2d - 45abc + 40b^3. \end{aligned}$$

An *orthogonal reciprocant* may be defined as a mixed reciprocant whose form remains invariable (save as to the acquisition of an extraneous factor when the reciprocant is not absolute) when any orthogonal substitution is impressed on the variables  $x$  and  $y$ . Concerning such reciprocants, we have the very

beautiful theorem: *If  $R$  and  $\frac{dR}{dt}$  are both of them reciprocants, then  $R$  is an orthogonal reciprocant.*

First suppose  $R$  to be an absolute reciprocant; *i. e.* let

$$R = qR' \quad (q = \pm 1),$$

where  $R$  is a function of  $t, a, b, c, \dots$  and  $R'$  the same function of  $\tau, \alpha, \beta, \gamma, \dots$ ; then, denoting by  $\Delta R$  the variation of  $R$  due to the variation of  $y$  by  $\varepsilon x$ , and by  $DR$  the variation of  $R$  due to the variation of  $x$  by  $-\varepsilon y$ , we have

$$\Delta R = \varepsilon \frac{dR}{dt}.$$

For the variation of  $t$  is  $\varepsilon$  and the variations of  $a, b, c, \dots$  vanish. Similarly

$$DR' = -\varepsilon \frac{dR'}{d\tau}.$$

Now, since

$$\begin{aligned} R &= qR', \\ DR &= qDR' = -\varepsilon q \frac{dR'}{d\tau}, \end{aligned}$$

therefore

$$DR + \Delta R = \varepsilon \left( \frac{dR}{dt} - q \frac{dR'}{d\tau} \right);$$

*i. e.* the total variation of  $R$  (due to the change of  $x$  into  $x - \varepsilon y$  and of  $y$  into  $y + \varepsilon x$ ) vanishes if

$$\frac{dR}{dt} = q \frac{dR'}{d\tau}.$$

Hence, if  $R$  be an absolute orthogonal reciprocant,  $\frac{dR}{dt}$  is also an absolute reciprocant (though it is not orthogonal) of the same character as  $R$ .

If  $R$  be not absolute, suppose its characteristic to be  $\mu$ ; then it can be made absolute by dividing it by  $a^{\frac{\mu}{3}}$ . The application of the foregoing method of variations will now prove that  $\frac{d}{dt} \left( \frac{R}{a^{\frac{\mu}{3}}} \right)$  is an absolute reciprocant of the same character as  $\frac{R}{a^{\frac{\mu}{3}}}$ . But  $\frac{d}{dt} \left( \frac{R}{a^{\frac{\mu}{3}}} \right) = \frac{1}{a^{\frac{\mu}{3}}} \frac{dR}{dt}$ . Hence  $\frac{dR}{dt}$  is a reciprocant whose characteristic is  $\mu$ , and character the same as that of  $R$ .

The simplest Orthogonal Reciprocant is the form

$$(1 + t^2)b - 3a^2t,$$

which occurs on p. 19 of Boole's Differential Equations. When equated to zero it is the general differential equation of a circle. It is noticeable that although Boole obtains this form by equating to zero the differential of the radius of curvature

$$\frac{(1 + t^2)^{\frac{3}{2}}}{a},$$

he does not recognize the fact that it vanishes at points of maximum or minimum curvature of any plane curve, but says that the "geometrical property which this equation expresses is the invariability of the radius of curvature."

Taking this form as an example of our general theorem, let

$$R = (1 + t^2)b - 3a^2t;$$

then

$$\frac{dR}{dt} = 2bt - 3a^2,$$

which is the familiar Schwarzian. Observe that

$$(1 + t^2)b - 3a^2t = -t^6\{(1 + \tau^2)\beta - 3\alpha^2\tau\}$$

and

$$2bt - 3a^2 = -t^6(2\beta\tau - 3\alpha^2),$$

so that the characteristic and character are the same for both these forms.

The form  $ct - 5ab$ , which we have called the Post-Schwarzian, when multiplied by 2 and integrated with respect to  $t$ , gives

$$ct^2 - 10abt + \phi(a, b, \dots).$$

In order that this may be a reciprocant, we must have

$$\phi(a, b, \dots) = c + 15a^3.$$

In this way the Orthogonal Reciprocant

$$(1 + t^2)c - 10abt + 15a^3$$

was obtained originally.

It will be easy to verify that this is a reciprocant by means of the identical relations

$$\begin{aligned} t &= \frac{1}{\tau}, \\ a &= -\frac{\alpha}{\tau^3}, \\ b &= -\frac{\beta\tau - 3\alpha^2}{\tau^5}, \\ c &= -\frac{\gamma\tau^3 - 10\alpha\beta\tau + 15\alpha^3}{\tau^7}. \end{aligned}$$

We shall find that

$$(1 + t^2)c - 10abt + 15a^3 = -t^7\{(1 + \tau^2)\gamma - 10\alpha\beta\tau + 15\alpha^3\},$$

and comparing this with

$$ct - 5ab = -t^7(\gamma\tau - 5\alpha\beta),$$

it will be noticed that both forms have the same character and the same characteristic.

The complete primitive of the differential equation

$$c(1 + t^2) - 10abt + 15a^3 = 0$$

has been found by Mr. Hammond and Prof. Greenhill. The solution may be written in the following forms :

$$\left. \begin{aligned} x &= \int \frac{dt}{\sqrt{x(1-15t^2+15t^4-t^6)+\lambda(6t-20t^3+6t^5)}} + \mu \\ y &= \int \frac{tdt}{\sqrt{x(1-15t^2+15t^4-t^6)+\lambda(6t-20t^3+6t^5)}} + \nu \end{aligned} \right\},$$

$$\left. \begin{aligned} x &= \int \frac{\cos(\theta-A)d\theta}{\sqrt{B \cos 6(\theta-A)}} + \text{const.} \\ y &= \int \frac{\sin(\theta-A)d\theta}{\sqrt{B \cos 6(\theta-A)}} + \text{const.} \end{aligned} \right\}.$$

where

$$k^{12} \text{tn}^2(X, k) = k'^2 \text{tn}^2(Y, k'),$$

and

$$k = \sin 15^\circ, \quad k' = \sin 75^\circ,$$

$$X = lx + my + n_1,$$

$$Y = mx - ly + n_2,$$

$l, m, n_1, n_2$  being arbitrary constants.

The last two forms of solution are due to Prof. Greenhill.

## LECTURE VII.

I have frequently referred to, and occasionally dilated on, the analogy between pure reciprocants and invariants. A new bond of connection between the two theories has been established by Capt. MacMahon, which I will now explain. Let me, by way of preface, so far anticipate what I shall have to say on the Theorem of Aggregation in Invariants (*i. e.* the theorem concerning the number of linearly independent invariants of a given type) as to remark that the proof of this theorem, first given by me in *Crelle's Journal* and subsequently in the *Phil. Mag.* for March, 1878, depends on the fact that if we take two operators, viz. the Annihilator, say

$$\Omega = a_0 \partial_{a_1} + 2a_1 \partial_{a_2} + 3a_2 \partial_{a_3} + \dots + j a_{j-1} \partial_{a_j}$$

and its opposite, say

$$O = a_j \partial_{a_{j-1}} + 2a_{j-1} \partial_{a_{j-2}} + 3a_{j-2} \partial_{a_{j-3}} + \dots + j a_1 \partial_{a_0},$$

then  $(\Omega O - O\Omega)I$  is a multiple of  $I$ .

Thus, if  $I$  stands for any invariant (*i. e.* if  $\Omega I = 0$ ), it follows immediately that  $\Omega OI$  is a multiple of  $I$ , and consequently  $\Omega^m O^m I$  is also a multiple of  $I$ . We may call  $\Omega$  and  $O$ , which are exact opposites to each other, reversing operators.

Now, MacMahon has found out the reversor to  $V$ , the Annihilator of pure reciprocants. His reversing operator is no longer of a similar, though opposite, form to  $V$ , as  $O$  is to  $\Omega$ , but is simply  $\frac{d}{dx}$ ; nor is the effect of operating with  $V \frac{d}{dx}$  on any pure reciprocant  $R$  equivalent to multiplication by a merely numerical factor, as was the case with  $\Omega OI$ , but  $\left(V \frac{d}{dx}\right) R$  is a numerical multiple of  $aR$ , and as a consequence of this  $\left(V^m \frac{d^m}{dx^m}\right) R$  is a numerical multiple of  $a^m R$ . Thus the parallelism is like that between the two sexes, the same with a difference, as is usually the case in comparing the two theories.

This remarkable relation between the operators  $V$  and  $\frac{d}{dx}$  may be seen *a priori* if we assume that (as we shall hereafter prove) to each pure reciprocant  $R$  there is an annihilator  $V$  of the form

$$3a^2 \partial_b + (\dots) \partial_c + (\dots) \partial_d + (\dots) \partial_e + \dots$$

not containing  $\partial_a$  and linear in the remaining differential operators  $\partial_b, \partial_c, \partial_d, \dots$ . For if we call the characteristic  $\mu$ , by differentiating the absolute pure reciprocant  $\frac{R}{a^{\frac{\mu}{2}}}$  with respect to  $x$  we obtain, as was shown in the last lecture, the pure reciprocant

$$3a \frac{dR}{dx} - \mu b R.$$

Since this is annihilated by  $V$ , we have

$$3a \left(V \frac{d}{dx}\right) R - \mu R Vb - \mu b VR = 0.$$

But, since  $R$  is a pure reciprocant,  $VR = 0$ ; and from the assumed form of  $V$  it follows that

$$Vb = 3a^2.$$

Hence

$$3a \left(V \frac{d}{dx}\right) R - 3\mu a^2 R = 0,$$

or

$$\left(V \frac{d}{dx}\right) R = \mu a R.$$

Thus the operation of  $V \frac{d}{dx}$  is equivalent to multiplication by  $\mu a$ , so that (barring the introduction of  $a$ )  $V$  restores to  $\frac{dR}{dx}$  the form it had antecedent to the operation of  $\frac{d}{dx}$ , and may be called a qualified reversor to  $\frac{d}{dx}$ .

For example, suppose that

$$R = 3ac - 5b^2.$$

Since we are using *natural* letters for the derivatives of  $y$  with respect to  $x$ , we have

$$\frac{d}{dx} = b\partial_a + c\partial_b + d\partial_c + \dots,$$

and, as we shall presently see,

$$V = 3a^2\partial_b + 10ab\partial_c + (15ac + 10b^2)\partial_a + \dots$$

$$\text{Now, } \frac{dR}{dx} = (b\partial_a + c\partial_b + d\partial_c)(3ac - 5b^2) = 3bc - 10bc + 3ad = 3ad - 7bc.$$

Operating on this with  $V$ , we find

$$V \frac{dR}{dx} = V(3ad - 7bc) = -21a^2c - 70ab^2 + 3a(15ac + 10b^2) = 24a^2c - 40ab^2;$$

$$\text{i. e. } V \frac{d}{dx} (3ac - 5b^2) = 8a(3ac - 5b^2).$$

Let us now inquire whether it is possible so to determine an operator  $V$  that the relation

$$\left( V \frac{d}{dx} - \frac{d}{dx} V \right) F = (3i + w) a F$$

may be satisfied identically when  $F$  is any homogeneous isobaric function of the letters  $a, b, c, \dots$  of degree  $i$  and weight  $w$ . If so, we must be able to satisfy each of the equations

$$\begin{aligned} \left( V \frac{d}{dx} - \frac{d}{dx} V \right) a &= 3a^2, \\ \left( V \frac{d}{dx} - \frac{d}{dx} V \right) b &= 4ab, \\ \left( V \frac{d}{dx} - \frac{d}{dx} V \right) c &= 5ac, \\ \left( V \frac{d}{dx} - \frac{d}{dx} V \right) d &= 6ad, \\ &\dots \end{aligned}$$

which are found by writing  $a, b, c, d, \dots$  successively in the place of  $F$ .

Now  $\frac{da}{dx} = b$ ,  $\frac{db}{dx} = c$ ,  $\frac{dc}{dx} = d$ , . . . . so that the above equations may be written

$$Vb = 3a^2 + \frac{d}{dx}(Va),$$

$$Vc = 4ab + \frac{d}{dx}(Vb),$$

$$Vd = 5ac + \frac{d}{dx}(Vc),$$

$$Ve = 6ad + \frac{d}{dx}(Vd),$$

$$\dots\dots\dots$$

These equations are sufficient to completely determine  $V$  on the supposition previously made that it is linear in the differential operators and does not contain  $\partial_a$ ; for, since  $V$  is linear, it must be of the form

$$(Va)\partial_a + (Vb)\partial_b + (Vc)\partial_c + \dots,$$

and, since it does not contain  $\partial_a$ , we must have  $Va = 0$ , and therefore

$$Vb = 3a^2,$$

$$Vc = 4ab + \frac{d}{dx}(3a^2) = 4ab + 6ab = 10ab,$$

$$Vd = 5ac + \frac{d}{dx}(10ab) = 5ac + 10b^2 + 10ac = 15ac + 10b^2,$$

$$Ve = 6ad + \frac{d}{dx}(15ac + 10b^2) = 6ad + 15bc + 20bc + 15ad = 21ad + 35bc,$$

$$\dots\dots\dots$$

Hence  $V = 3a^2\partial_b + 10ab\partial_c + (15ac + 10b^2)\partial_a + (21ad + 35bc)\partial_e + \dots$

When the modified letters  $a_0, a_1, a_2, \dots$  are used, we shall have, in consequence of the change of notation,  $\left(V\frac{d}{dx}\right)R = 2\mu a_0 R$  (instead of  $\mu a R$ ). If, as before, we seek to satisfy the equation

$$\left(V\frac{d}{dx} - \frac{d}{dx}V\right)F = 2(3i + w)a_0F, \quad (1)$$

we shall find, on writing  $a_n$  in the place of  $F$ ,

$$\left(V\frac{d}{dx} - \frac{d}{dx}V\right)a_n = 2(3 + n)a_0a_n. \quad (2)$$

This condition will be sufficient, as well as necessary, for the satisfaction of (1) when  $V$  is linear; for then

$$V\frac{d}{dx} - \frac{d}{dx}V$$

will also be linear, its general term being

$$\left( V \frac{da_n}{dx} - \frac{d}{dx} V a_n \right) \partial_{a_n},$$

which is equal to  $2(3+n)a_0a_n\partial_{a_n}$  by equation (2). Hence

$$\begin{aligned} \left( V \frac{d}{dx} - \frac{d}{dx} V \right) F &= \text{a sum of terms of the form } 2(3+n)a_0a_n\partial_{a_n} F \\ &= 2a_0(3a_0\partial_{a_0} + 3a_1\partial_{a_1} + 3a_2\partial_{a_2} + \dots) F \\ &\quad + 2a_0(a_1\partial_{a_1} + 2a_2\partial_{a_2} + \dots) F; \end{aligned}$$

i. e. equation (1) is satisfied whenever (2) is. Writing in (2)

$$\frac{da_n}{dx} = (n+3)a_{n+1},$$

$$\text{we obtain} \quad (n+3) V a_{n+1} = 2(n+3)a_0a_n + \frac{d}{dx} (V a_n), \quad (3)$$

from which the values of  $V a_n$  may be successively determined.

When  $V a_0 = 0$ , the value of  $V a_n$ , which satisfies (3), is

$$V a_n = \frac{n+3}{2} (a_0a_{n-1} + a_1a_{n-2} + \dots + a_{n-2}a_1 + a_{n-1}a_0);$$

$$\text{thus} \quad V a_1 = \frac{4}{2} \cdot a_0^2, \quad V a_2 = 5a_0a_1, \quad V a_3 = 6a_0a_2 + 3a_1^2, \dots$$

and the value of  $V$  is therefore

$$\frac{4}{2} \cdot a_0^2 \partial_{a_1} + 5a_0a_1 \partial_{a_2} + 6 \left( a_0a_2 + \frac{1}{2} a_1^2 \right) \partial_{a_3} + 7(a_0a_3 + a_1a_2) \partial_{a_4} + \dots$$

Now that we are on the subject of parallelism between the old and new worlds of Algebraical Form, I feel tempted to point out yet another very interesting bond of connection between them. There is a theorem concerning Invariants which I am not aware that any one but myself has noticed, or at all events I do not remember ever seeing it in print,\* which is this: If we take any "invariant" and regard its most advanced letter as a variable, or say rather as the ratio of two variables  $u:v$ , by multiplying by a proper power of  $v$  we obtain a new Quantic in  $u, v$ ; or, if we take any number of such invariants with the same most advanced letter (or, as we may call it in a double sense, the same radical letter) in common, we shall have a system of binary Quantics in  $u, v$ . My theorem is, or was, that an Invariant of any one or more of such Quantics is an Invariant of the original Quantic. I recently found a similar proposition to be true for Reciprocants, viz. forming as before a system of *pure*

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\* The theorem is, however, given in Vol. XI, p. 98 of the Bulletin de la Société Mathématique de France, in a paper by M. Perrin, which has only recently come under the lecturer's notice.

Reciprocants into Quantics in  $u, v$ , any "Invariant" of such system is itself a Reciprocant.

The two theorems may be stated symbolically thus :

$$\left. \begin{array}{l} II' = I'' \\ IR = R' \end{array} \right\}.$$

On mentioning this to Mr. L. J. Rogers, he sent me next day a proof which, although only stated as applicable to Reciprocants, is equally so, *mutatis mutandis*, to Invariants. Although given for a single invariant, it applies equally to a system.

I give Mr. Rogers' proof that any invariant of a *pure* reciprocant (the proof will not hold for impure ones) is a pure reciprocant; or rather I use his method to prove the analogous theorem that any invariant of an invariant is itself an invariant. It will be seen hereafter that this same proof applies to *pure* reciprocants with only trifling changes; but the proof as given by Mr. Rogers requires some further considerations to be gone into for which we are not yet ripe.

Consider, for the sake of simplicity, the binary Quintic

$$(a, b, c, d, e, f | x, y)^5,$$

and let  $I$  be any invariant of it (satisfied or unsatisfied); then

$$I = a_0 f^n + a_1 f^{n-1} + a_2 f^{n-2} + \dots + a_n,$$

where  $a_0, a_1, a_2, \dots, a_n$  do not contain  $f$ , but are functions of  $a, b, c, d, e$  alone.

Let the Protomorphs for our Quintic be denoted by  $A, B, C, D, E, F$ ; then

$$F = a^2 f - 5abe + 2acd + 8b^2 d + 6bc^2.$$

Eliminating  $f$  from  $I$  by means of this equation, we have

$$Ia^{2n} = A_0 F^n + A_1 F^{n-1} + A_2 F^{n-2} + \dots + A_n,$$

where  $A_0, A_1, A_2, \dots, A_n$  are all of them invariants (not necessarily integral forms, but this is immaterial to the proof, for  $\Omega$  annihilates fractional and integral invariants alike). For

$$\Omega(Ia^{2n}) = \Omega(A_0 F^n + A_1 F^{n-1} + \dots + A_n),$$

and, in consequence of  $Ia^{2n}$  and  $F$  being invariants, so that, as regards  $\Omega$ ,  $F$  may be treated as if it were a constant, this becomes

$$0 = F^n \Omega A_0 + F^{n-1} \Omega A_1 + F^{n-2} \Omega A_2 + \dots + \Omega A_n,$$

in which the coefficients of the several powers of  $F$  must be separately equated

to zero. In other words,  $A_0, A_1, A_2, \dots, A_n$  are all of them invariants. Now, any invariant of

$$A_0 F^n + A_1 F^{n-1} + A_2 F^{n-2} + \dots + A_n$$

is a function of  $A_0, A_1, A_2, \dots, A_n$ , and therefore an invariant.

(N. B.—We cannot assume that any function of general reciprocants is itself a reciprocant.)

Again, since

$$A_0 F^n + \dots + A_n, \text{ and } a_0 f^n + \dots + a_n$$

are connected by the substitution

$$F = a^2 f - 5abe + \dots,$$

which is *linear* in respect to the letters  $F$  and  $f$ , any invariant of

$$A_0 F^n + \dots + A_n$$

is (to a factor *près*, that factor being a power of  $a$  which is itself an invariant) equal to the corresponding invariant of

$$a_0 f^n + \dots + a_n.$$

But every invariant of the former has been shown to be an invariant of the original quantic, and therefore every invariant of the latter is so also.

I add some examples in illustration of this theorem:

Ex. 1. Take the invariant of the Quintic

$$\begin{aligned} a^2 f^2 - 10abef + 4acdf + 16b^2 df - 12bc^2 f + 16ace^2 + 9b^2 e^2 - 12ad^2 e - 76bcde \\ + 48c^3 e + 48bd^3 - 32c^2 d^2. \end{aligned}$$

The discriminant of this, considered as a quadratic in  $f$ , is

$$\begin{aligned} a^2 (16ace^2 + 9b^2 e^2 - 12ad^2 e - 76bcde + 48c^3 e + 48bd^3 - 32c^2 d^2) \\ - (5abe - 2acd - 8b^2 d + 6bc^2)^2 \\ = 16a^3 ce^2 - 16a^2 b^2 e^2 - 12a^3 d^2 e - 56a^2 bcde + 48a^2 c^3 e + 80ab^3 de - 60ab^2 c^2 e + 48a^2 bd^3 \\ - 36a^2 c^2 d^2 - 32ab^3 cd^2 - 64b^4 d^2 + 24abc^3 d + 96b^3 c^2 d - 36b^2 c^4. \end{aligned}$$

It will be found on trial that this is divisible by the invariant

$$4(ae - 4bd + 3c^2),$$

the quotient being

$$\begin{aligned} 4a^2 ce - 4ab^2 e - 3a^2 d^2 + 2abcd + 4b^3 d - 3b^3 c^2 \\ = 3a(ace - b^2 e - ad^2 + 2bcd - c^3) + (ac - b^2)(ae - 4bd + 3c^2). \end{aligned}$$

Thus the discriminant of the quadratic in  $f$ , *i. e.* of the invariant

$$a^2 f^2 - 2f(5abe - 2acd + 8b^2 d - 6bc^2) + \dots,$$

is shown to be an invariant. It will further illustrate the proof of the theorem

if we remark that precisely the same invariant is obtained by eliminating  $f$  between the above form and the protomorph

$$a^2f - 5abe + 2acd + 8b^2d - 6bc^2.$$

Ex. 2. If we take the pure reciprocant

$$45a^3d^2 - 450a^2bcd + 400ab^3d + 192a^2c^3 + 165ab^2c^2 - 400b^4c,$$

which, from its similarity to the Discriminant of the Cubic, I have called the Quasi-Discriminant, and form *its* discriminant, when regarded as a quadratic in  $d$ , we find

$$45a^3(192a^2c^3 + 165ab^2c^2 - 400b^4c) - (225a^2bc - 200ab^3)^2.$$

If, in this expression, we write  $P = 3ac - 5b^2$ , so that  $3ac = P + 5b^2$ , it becomes  $5.64a^2(P + 5b^2)^3 + 5.165a^2b^2(P + 5b^2)^2 - 15.400a^2b^4(P + 5b^2) - 625a^2b^2(3P + 7b^2)^2$ . On performing the calculation it will be found that all the terms involving  $b$  will disappear from this result, and there will remain the single term  $320a^2P^3$ , *i. e.*  $320a^2(3ac - 5b^2)^3$ , which is a reciprocant.

## LECTURE VIII.

In my last lecture the complete expression, both in terms of the modified and unmodified letters, was obtained for  $V$ , the annihilator for pure reciprocants assuming its existence and its form. These assumptions I shall now make good by proving, from first principles, the fundamental theorem that the satisfaction of the equation

$$VR = 0$$

is a necessary and sufficient condition in order that  $R$  may be a pure reciprocant.

It will be advantageous to use the modified system of letters, in which

$$t, a_0, a_1, a_2, \dots \text{ stand for } \frac{dy}{dx}, \frac{1}{1.2} \cdot \frac{d^2y}{dx^2}, \frac{1}{1.2.3} \cdot \frac{d^3y}{dx^3}, \frac{1}{1.2.3.4} \cdot \frac{d^4y}{dx^4}, \dots$$

$$\text{and } \alpha_0, \alpha_1, \alpha_2, \dots \text{ for } \frac{1}{1.2} \cdot \frac{d^2x}{dy^2}, \frac{1}{1.2.3} \cdot \frac{d^3x}{dy^3}, \frac{1}{1.2.3.4} \cdot \frac{d^4x}{dy^4}, \dots$$

respectively. Let the variation due to the change of  $x$  into  $x + \varepsilon y$ , where  $\varepsilon$  is an infinitesimal number, be denoted by  $\Delta$ . Obviously this change leaves the value of each of the quantities  $\alpha_0, \alpha_1, \alpha_2, \dots$  unaltered, and therefore

$$\Delta R(\alpha_0, \alpha_1, \alpha_2, \dots) = 0,$$

whatever the nature of  $R$  may be. But when  $R$  is a pure reciprocant,

$$R(a_0, a_1, a_2, \dots) = \pm t^\mu R(a_0, a_1, a_2, \dots),$$

whence it immediately follows that

$$\Delta t^{-\mu} R(a_0, a_1, a_2, \dots) = 0.*$$

Before proceeding to determine the values of

$$\Delta t, \Delta a_0, \Delta a_1, \Delta a_2, \dots$$

it will be useful to remark that since

$$\frac{dy}{dx} = t, \frac{d^2y}{dx^2} = 1.2.a_0, \frac{d^3y}{dx^3} = 1.2.3.a_1, \dots$$

we have 
$$\frac{dt}{dx} = 2a_0, \frac{da_0}{dx} = 3a_1, \dots$$

and generally 
$$\frac{da_n}{dx} = (n+3)a_{n+1}.$$

Now let  $[t]$  denote the augmented value of  $t$ , and in general let  $[ ]$  be used to signify that the augmented value of the quantity enclosed in it is to be taken. Then

$$\begin{aligned} [t] &= \frac{dy}{d[x]} = \frac{dy}{d(x + \epsilon y)} = \frac{dy}{dx \left(1 + \epsilon \frac{dy}{dx}\right)} = \frac{t}{1 + \epsilon t} \\ &= t - \epsilon t^2; \end{aligned}$$

so also 
$$\begin{aligned} 2[a_0] &= [2a_0] = \frac{d[t]}{d[x]} = \frac{d[t]}{d(x + \epsilon y)} = \frac{d[t]}{dx(1 + \epsilon t)} = (1 - \epsilon t) \frac{d[t]}{dx} \\ &= (1 - \epsilon t) \frac{d}{dx} (t - \epsilon t^2) = (1 - \epsilon t)(2a_0 - 4\epsilon ta_0) \\ &= 2a_0 - 6\epsilon ta_0; \end{aligned}$$

i. e. 
$$[a_0] = a_0 - 3\epsilon ta_0.$$

Reasoning precisely similar to that which gave

$$2[a_0] = (1 - \epsilon t) \frac{d}{dx} [t]$$

leads to the formula

$$(n+3)[a_{n+1}] = (1 - \epsilon t) \frac{d}{dx} [a_n],$$

from which the augmented values of  $a_1, a_2, a_3, \dots$  may be found by giving to  $n$  the values  $0, 1, 2, \dots$  in succession. Thus, writing  $n = 0$ , we have

\* It has been suggested by Mr. J. Chevallier that the proof might be simplified by considering the variation  $\Delta a_0^{-\frac{\mu}{3}} R(a_0, a_1, a_2, \dots)$  instead of  $\Delta t^{-\mu} R(a_0, a_1, a_2, \dots)$ .

$$\begin{aligned} 3 [a_1] &= (1 - \epsilon t) \frac{d}{dx} [a_0] = (1 - \epsilon t) \frac{d}{dx} (a_0 - 3\epsilon t a_0) \\ &= (1 - \epsilon t)(3a_1 - 9\epsilon t a_1 - 6\epsilon a_0^2) = 3a_1 - \epsilon(12ta_1 + 6a_0^2), \end{aligned}$$

or  $[a_1] = a_1 - \epsilon(4ta_1 + 2a_0^2).$

Similarly, when  $n = 1$ ,

$$\begin{aligned} 4 [a_2] &= (1 - \epsilon t) \frac{d}{dx} [a_1] = (1 - \epsilon t) \frac{d}{dx} (a_1 - 4\epsilon t a_1 - 2\epsilon a_0^2) \\ &= (1 - \epsilon t)(4a_2 - 16\epsilon t a_2 - 20\epsilon a_0 a_1) \\ &= 4a_2 - 20\epsilon t a_2 - 20\epsilon a_0 a_1, \end{aligned}$$

and  $[a_2] = a_2 - 5\epsilon(ta_2 + a_0 a_1).$

Again, 
$$\begin{aligned} 5 [a_3] &= (1 - \epsilon t) \frac{d}{dx} [a_2] = (1 - \epsilon t) \frac{d}{dx} (a_2 - 5\epsilon t a_2 - 5\epsilon a_0 a_1) \\ &= (1 - \epsilon t)(5a_3 - 25\epsilon t a_3 - 30\epsilon a_0 a_2 - 15\epsilon a_1^2) \\ &= 5a_3 - 30\epsilon t a_3 - 30\epsilon a_0 a_2 - 15\epsilon a_1^2, \end{aligned}$$

so that  $[a_3] = a_3 - \epsilon(6ta_3 + 6a_0 a_2 + 3a_1^2).$

In like manner we shall find

$$[a_4] = a_4 - 7\epsilon(ta_4 + a_0 a_3 + a_1 a_2).$$

These results may be written in a more symmetrical form ; thus :

$$\begin{aligned} 2 [t] &= 2t - 2\epsilon t^2, \\ 2 [a_0] &= 2a_0 - 3\epsilon(ta_0 + a_0 t), \\ 2 [a_1] &= 2a_1 - 4\epsilon(ta_1 + a_0^2 + a_1 t), \\ 2 [a_2] &= 2a_2 - 5\epsilon(ta_2 + a_0 a_1 + a_1 a_0 + a_2 t), \\ 2 [a_3] &= 2a_3 - 6\epsilon(ta_3 + a_0 a_2 + a_1^2 + a_2 a_0 + a_3 t), \\ 2 [a_4] &= 2a_4 - 7\epsilon(ta_4 + a_0 a_3 + a_1 a_2 + a_2 a_1 + a_3 a_0 + a_4 t). \end{aligned}$$

The general law

$$2 [a_n] = 2a_n - (n + 3) \epsilon(ta_n + a_0 a_{n-1} + \dots + a_{n-1} a_0 + a_n t),$$

or, as it may also be written,

$$\Delta a_n = -\frac{n+3}{2} \epsilon(ta_n + a_0 a_{n-1} + \dots + a_{n-1} a_0 + a_n t),$$

admits of an easy inductive proof.

Assuming the truth of the theorem for  $[a_n]$ , and writing for brevity in what follows,

$$S_n = ta_n + a_0 a_{n-1} + a_1 a_{n-2} + \dots + a_{n-2} a_1 + a_{n-1} a_0 + a_n t,$$

we have  $[a_n] = a_n - \frac{n+3}{2} \epsilon S_n.$

Now,

$$\begin{aligned}
 \frac{dS_n}{dx} &= (n+3)ta_{n+1} + 2a_0a_n \\
 &\quad + (n+2)a_0a_n + 3a_1a_{n-1} \\
 &\quad + (n+1)a_1a_{n-1} + 4a_2a_{n-2} \\
 &\quad + \dots + \dots \\
 &\quad + 4a_{n-2}a_2 + (n+1)a_{n-1}a_1 \\
 &\quad + 3a_{n-1}a_1 + (n+2)a_na_0 \\
 &\quad + 2a_na_0 + (n+3)a_{n+1}t \\
 &= (n+4)(ta_{n+1} + a_0a_n + a_1a_{n-1} + \dots + a_{n-1}a_1 + a_na_0 + a_{n+1}t) - 2ta_{n+1} \\
 &= (n+4)S_{n+1} - 2ta_{n+1}.
 \end{aligned}$$

Hence  $\frac{d}{dx} [a_n] = (n+3)a_{n+1} - \frac{n+3}{2}\varepsilon\{(n+4)S_{n+1} - 2ta_{n+1}\}.$

But, as we have already seen,

$$(n+3)[a_{n+1}] = (1-\varepsilon t) \frac{d}{dx} [a_n];$$

consequently,

$$[a_{n+1}] = (1-\varepsilon t)a_{n+1} - \frac{n+4}{2}\varepsilon S_{n+1} + \varepsilon ta_{n+1} = a_{n+1} - \frac{n+4}{2}\varepsilon S_{n+1};$$

*i. e.* the theorem holds for  $[a_{n+1}]$  when it holds for  $[a_n]$ . But we know that it is true for the cases  $n = 0, 1, 2, 3, 4$ , and therefore it is true universally.

Resuming the proof of the main theorem, it has been shown that

$$\Delta t^{-\mu} R(a_0, a_1, a_2, \dots) = 0;$$

*i. e.*  $-\mu t^{-1} \Delta t + R^{-1} \Delta R = 0,$

or  $-\mu R t^{-1} \Delta t + \frac{dR}{da_0} \Delta a_0 + \frac{dR}{da_1} \Delta a_1 + \frac{dR}{da_2} \Delta a_2 + \dots = 0.$

But

$$\begin{aligned}
 \Delta t &= -\varepsilon t^2, \\
 \Delta a_0 &= -3\varepsilon t a_0, \\
 \Delta a_1 &= -\varepsilon (4t a_1 + 2a_0^2), \\
 \Delta a_2 &= -\varepsilon (5t a_2 + 5a_0 a_1), \\
 \Delta a_3 &= -\varepsilon (6t a_3 + 6a_0 a_2 + 3a_1^2), \\
 \Delta a_4 &= -\varepsilon (7t a_4 + 7a_0 a_3 + 7a_1 a_2), \\
 &\dots
 \end{aligned}$$

and consequently

$$\begin{aligned}
 &t(\mu - 3a_0\partial_{a_0} - 4a_1\partial_{a_1} - 5a_2\partial_{a_2} - 6a_3\partial_{a_3} - 7a_4\partial_{a_4} - \dots)R \\
 &- \left\{ 4\left(\frac{a_0^2}{2}\right)\partial_{a_1} + 5(a_0a_1)\partial_{a_2} + 6\left(a_0a_2 + \frac{a_1^2}{2}\right)\partial_{a_3} + 7(a_0a_3 + a_1a_2)\partial_{a_4} + \dots \right\} R = 0.
 \end{aligned}$$

This is equivalent to the two conditions

$$(3a_0\partial_{a_0} + 4a_1\partial_{a_1} + 5a_2\partial_{a_2} + \dots)R = \mu R$$

and

$$VR = 0,$$

where

$$V = 4\left(\frac{a_0^2}{2}\right)\partial_{a_1} + 5(a_0a_1)\partial_{a_2} + 6\left(a_0a_2 + \frac{a_1^2}{2}\right)\partial_{a_3} + 7(a_0a_3 + a_1a_2)\partial_{a_4} + \dots$$

For greater simplicity I confine what I have to say to the only essential case, to which every other may be reduced, of a *homogeneous* pure reciprocant. The equation

$$(3a_0\partial_{a_0} + 4a_1\partial_{a_1} + 5a_2\partial_{a_2} + \dots)R = \mu R$$

shows that for every term  $w + 3i$  is constant; *i. e.*  $w$  is constant and therefore the function  $R$  is isobaric. This is also immediately deducible from the form of the relations between  $a_0, a_1, a_2, \dots$ ;  $\alpha_0, \alpha_1, \alpha_2, \dots$ , and, what is important to notice, for future purposes,

$$F(a_0, a_1, a_2, \dots) - t^\mu F(\alpha_0, \alpha_1, \alpha_2, \dots),$$

when  $F$  is a homogeneous isobaric function, and  $\mu = w + 3i$  is itself a homogeneous function of  $(a_0, a_1, a_2, \dots)$ , whose degree is the same as that of  $F$ .

The only condition affecting  $R$ , a function of  $a_0, a_1, a_2, \dots$ , supposed homogeneous and isobaric, is

$$VR = 0.$$

I shall now prove the converse, that if  $R = F(a_0, a_1, a_2, \dots)$  (being homogeneous and isobaric) has  $V$  for its annihilator, then  $R$  is a pure reciprocant. Let  $D$  be the value of  $F(a_0, a_1, a_2, \dots) - t^\mu F(\alpha_0, \alpha_1, \alpha_2, \dots)$  expressed as a function of  $a_0, a_1, a_2, \dots$  alone. Then  $D$  will be a function of the same type as  $F(a_0, a_1, a_2, \dots)$ .

Suppose that  $\Delta D = 0$ ;

*i. e.* that the variation of  $D$  due to the change of  $x$  into  $x + \varepsilon y$  vanishes in virtue of the equation  $VR = 0$ .

Let  $D$  become  $D'$  when  $y$  receives an arbitrary variation  $y + \eta u$ , where  $\eta$  is an infinitesimal constant and  $u$  an arbitrary function of  $x$ ; then the variation of  $D'$  will vanish when  $x$  is changed into  $x + \varepsilon y + \varepsilon \eta u$ , and consequently when  $x$  is changed into  $x + \varepsilon y$  the variation of  $D'$  will also vanish. Hence

$$\Delta D' = 0,$$

and if we take the difference of the variations of  $D$  and  $D'$ , we shall find

$$\Delta\left(u''\frac{d}{da_0}D + u'''\frac{d}{da_1}D + u^{iv}\frac{d}{da_2}D + \dots\right) = 0.$$

Now, the arbitrary nature of the function  $u$  shows that we must have

$$\Delta \frac{d}{da_0} D = 0, \Delta \frac{d}{da_1} D = 0, \Delta \frac{d}{da_2} D = 0, \dots$$

and if we reason on  $\frac{d}{da_0} D, \frac{d}{da_1} D, \dots$  in the same way as we have on  $D$ , we see that the variation  $\Delta$  of each of the second differential derivatives of  $D$  will also vanish; and, pursuing the same argument further, it will be evident that the  $\Delta$  of any derivative of  $D$ , of any order whatever, with respect to  $a_0, a_1, a_2, \dots$  will vanish. Hence  $D = 0$ ;

for if this is not so we may, supposing  $D$  to be a function of degree  $i$  in the letters  $a_0, a_1, a_2, \dots$ , take the  $\Delta$  of each of the differential derivatives of  $D$  of the order  $i - 1$ ; each of these variations would vanish by what precedes; *i. e.* the variation due to the change of  $x$  into  $x + \varepsilon y$  of each of the letters  $a_0, a_1, a_2, \dots$  contained in  $D$  would be identically zero, which is absurd. We see, therefore, that when  $\Delta D = 0$  (*i. e.* when  $R$  is annihilated by  $V$ ),  $D = 0$ , or

$$F(a_0, a_1, a_2, \dots) = {}^t F(a_0, a_1, a_2, \dots),$$

which proves the converse proposition.

It will not fail to be noticed how much language, and as a consequence algebraical thought (for words are the tools of thought), is facilitated by the use of the concept of annihilation in lieu of that of equality as expressed by a partial differential equation.

It is somewhat to the point that in the recent two grand determinations of the order of precedence among the so-called fixed stars relative to our planet, as approximately represented by the intensities of the light from them which reaches the eye, the one is directed by the principle of annihilation, the other by that of equality. Prof. Pritchard's method essentially consists in determining what relative thicknesses of an interposed glass screen, effected by means of a sliding wedge of glass, will serve to extinguish the light of a star; that employed by Prof. Pickering depends on finding what degree of rotation of an interposed prism of Iceland spar (a Nicol Prism) will serve to bring to an equality the ordinary image of one star with the extra-ordinary one of another. As these intensities depend on the squared sines and cosines of this angle of rotation measured from the position of non-visibility of one of them, it follows that the tangent squared of the twist measures the relative intensities by this method.

Hereafter it will be shown that if  $F$  is a homogeneous isobaric function of

$$y, y', y'', y''', \dots$$

whose weights are reckoned as

$$-2, -1, 0, 1, \dots$$

then, when  $x$  becomes  $x + hy$ , where  $h$  is any constant quantity,  $F$  becomes

$$(1 + ht)^{-\mu} e^{-\frac{hV_1}{1+ht}} F,$$

where  $t = y'$ ,  $V_1 = -t^2 \partial_t + V$ , and  $\mu = 3i + w$ ,  
 $i$  being the degree and  $w$  the weight of  $F$ .

From this, by an obvious course of reasoning, could be deduced as a particular case the condition of  $F(a_0, a_1, a_2, \dots)$  remaining a factor of its altered self when *any* linear substitutions are impressed on  $x$  and  $y$ ; viz. the necessary and sufficient condition is that  $F$  has  $V$  for its annihilator.

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## LECTURE IX.

The prerogative of a Pure Reciprocant is that it continues a factor of its altered self when the variables  $x$  and  $y$  are subjected to any linear substitution. Its form, like that of any other reciprocant, is of course persistent when the variables are interchanged; *i. e.* when in the general substitution, in which  $y$  is changed into

$$fy + gx + h$$

and  $x$  into

$$f'y + g'x + h',$$

we give the particular values  $h = 0$ ,  $h' = 0$ ,  $f = 0$ ,  $g' = 0$ ,  $f' = 1$ ,  $g = 1$ , to the constants. Stated geometrically, the theorem is that the evanescence of any pure reciprocant  $R$  indicates a property independent of transformation of axes in a plane. We suppose  $R$  to be homogeneous and isobaric in  $a, b, c, \dots$  (If it were not, the theorem could not hold, for either the change of  $y$  into  $xy$  or that of  $x$  into  $\lambda x$  would destroy the form.)

The persistence, under any linear substitution, of the form of pure reciprocants may be easily established as follows:

By a *semi-substitution* understand one where one of the variables remains unaltered. There are two such semi-substitutions, viz. where  $x$  remains unaltered, and where  $y$  does.

1°. Let  $x$  remain unaltered and  $y$  become  $fy + gx + h$ ; then  $a, b, c, \dots$  become  $fa, fb, fc, \dots$  respectively; and therefore

$$R(a, b, c, \dots) \text{ becomes } f^i R(a, b, c, \dots),$$

where  $i$  is the degree of  $R$ .

2°. Let  $y$  remain unchanged and  $x$  become  $f'y + g'x + h'$ . Then, instead of  $R$ , I look to its equal

$$\begin{aligned} &qt^\mu R(\alpha, \beta, \gamma, \dots) (q = \pm 1); \\ \text{i. e. to} & q\tau^{-\mu} R(\alpha, \beta, \gamma, \dots), \\ \text{which becomes} & q(f' + g'\tau)^{-\mu} g'^i R(\alpha, \beta, \gamma, \dots). \end{aligned}$$

Since  $R$  is a reciprocant, this is equal to

$$\frac{\tau^\mu}{(f' + g'\tau)^\mu} g'^i R(a, b, c, \dots),$$

or, replacing  $\tau$  by its equivalent  $\frac{1}{t}$ ,

$$(f't + g')^{-\mu} g'^i R(a, b, c, \dots).$$

Thus we see that the proposition is true for a semi-substitution of either kind. Consider now the complete substitution made by changing  $y$  into

$$\begin{aligned} &fy + gx + h \\ \text{and } x \text{ into} & Fy + Gx + H. \end{aligned}$$

If  $f = 0$  and  $G = 0$ , then  $\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \dots$  become  $\frac{g}{F^2} \cdot \frac{d^2x}{dy^2}, \frac{g}{F^3} \cdot \frac{d^3x}{dy^3}, \dots$ ; so that

$R(a, b, c, \dots)$  becomes  $\frac{g^i}{F^{2i+w}} \cdot R(\alpha, \beta, \gamma, \dots)$ ; and since this is equal to

$$\frac{g^i}{F^{2i+w}} \cdot qt^{-\mu} R(a, b, c, \dots),$$

the proposition is true.

But if either of the two letters  $f, G$  (say  $f$ ) is not zero, we may combine two semi-substitutions so as to obtain the complete substitution, in which  $y$  changes into

$$\begin{aligned} &fy + gx + h \\ \text{and } x \text{ changes into} & Fy + Gx + H. \end{aligned}$$

1°. Substitute  $y_1 (= fy + gx + h)$  for  $y$ , and  $x_1 (= x)$  for  $x$ .

2°. Then substitute  $y_2 (= y_1)$  for  $y_1$ , and  $x_2 (= f'y_1 + g'x_1 + h')$  for  $x_1$ .

By the first of these semi-substitutions

$$R(a, b, c, \dots)$$

takes up an extraneous factor  $f^t$ . By the second it acquires the factor

$$\left(f' \frac{dy_1}{dx_1} + g'\right)^{-\mu} g'^t, \text{ where } \frac{dy_1}{dx_1} = f \frac{dy}{dx} + g = ft + g.$$

Hence we see that the extraneous factor is a negative power of a linear function of  $t$ , which we shall presently particularize, though it is not essential to the present demonstration to do so.

It only remains to show how the combination of these two semi-substitutions can be made to give the complete one in question. We have

$$\begin{aligned} y_2 &= y_1 = fy + gx + h \\ \text{and } x_2 &= f'y_1 + g'x_1 + h' = f'(fy + gx + h) + g'x + h' \\ &= ff'y + (f'g + g')x + (f'h + h'). \end{aligned}$$

In order that this may be equal to  $Fy + Gx + H$ , we must be able to satisfy the equations

$$f' = \frac{F}{f}, \quad g' = G - \frac{gF}{f}, \quad h' = H - \frac{hF}{f},$$

which is always possible, since by hypothesis  $f$  is not zero. Similarly it may be shown that when  $f$  vanishes, but  $G$  does not, by substituting

$$\begin{aligned} 1^\circ. \quad x_1 & (= Fy + Gx + H) \text{ for } x, \text{ and } y_1 (= y) \text{ for } y, \\ 2^\circ. \quad x_2 & (= x_1) \text{ for } x_1, \text{ and } y_2 (= f''y_1 + g'x_1 + h'') \text{ for } y_1, \end{aligned}$$

we may so determine  $f''$ ,  $g''$ ,  $h''$  as to get the complete substitution as before.

In every case, therefore, any linear substitution impressed upon the variables  $x$  and  $y$  will leave  $R(a, b, c, \dots)$  unaltered, barring the acquisition of an extraneous factor which is a negative power of a linear function of  $t$ .

Now, the first semi-substitution introduces, as we have seen, the constant factor

$$f^t;$$

the second introduces the factor

$$\left(f' \frac{dy_1}{dx_1} + g'\right)^{-\mu} g'^t,$$

where

$$\frac{dy_1}{dx_1} = ft + g.$$

The complete extraneous factor is the product of these two, and is therefore

$$f^t g'^t (ff't + f'g + g')^{-\mu}.$$

To express  $f'$  and  $g'$  in terms of the constants of the complete substitution we have

$$f' = \frac{F}{f}, \quad g' = G - \frac{gF}{f}.$$

Writing these values for  $f'$  and  $g'$  in the expression just found, we obtain

$$(fG - gF)^i (Ft + G)^{-\mu},$$

which is the extraneous factor acquired by  $R$  when the complete substitution is made. For example, if  $x$  becomes

$$Fy + Gx + H$$

and  $y$  becomes

$$fy + gx + h,$$

the altered value of  $\alpha$  (i. e. of  $\frac{d^2y}{dx^2}$ ) is

$$(fG - gF)(Ft + G)^{-3}\alpha.$$

Corresponding to the simple interchange of the variables, we have

$$F = 1, \quad G = 0, \quad H = 0; \quad f = 0, \quad g = 1, \quad h = 0,$$

so that

$$fG - gF = -1,$$

and the altered value of  $\alpha$  is  $\frac{d^2x}{dy^2}$ ,

or

$$\alpha = -\frac{a}{t^3},$$

which is right. In this case the general value of the acquired extraneous factor

$$(fG - gF)^i (Ft + G)^{-\mu} \text{ becomes } (-)^i t^{-\mu},$$

thus showing, what we have already proved from other considerations, that the character of a pure reciprocant is odd or even according as its degree is odd or even.

We saw in the last lecture that *every* pure reciprocant necessarily satisfied the two conditions

$$(3a_0\partial_{a_0} + 4a_1\partial_{a_1} + 5a_2\partial_{a_2} + \dots) R = \mu R$$

(where  $\mu$  is the characteristic), and

$$VR = 0.$$

We also saw that  $VR = 0$  was a sufficient as well as necessary condition that *any homogeneous function*  $R$  of  $a_0, a_1, a_2, \dots$  should be a pure reciprocant. It will now be shown that every pure reciprocant is either homogeneous and isobaric, or else resolvable into a sum of homogeneous and isobaric reciprocants. Non-homogeneous mixed ones, it may be observed, are not so resolvable, so that the theorem only holds for pure reciprocants.

1°. Let us suppose that  $R$  (a pure reciprocant) is homogeneous in  $a_0, a_1, a_2, \dots$ ; then it must be isobaric also. For, if  $i$  is the degree of  $R$ , Euler's theorem shows that

$$(3a_0\partial_{a_0} + 3a_1\partial_{a_1} + 3a_2\partial_{a_2} + 3a_3\partial_{a_3} + \dots) R = 3iR;$$

and since  $R$  is a pure reciprocant, the condition

$$(3a_0\partial_{a_0} + 4a_1\partial_{a_1} + 5a_2\partial_{a_2} + 6a_3\partial_{a_3} + \dots) R = \mu R$$

is necessarily satisfied. Hence

$(a_1\partial_{a_1} + 2a_2\partial_{a_2} + 3a_3\partial_{a_3} + \dots) R = (\mu - 3i)R =$  a constant multiple of  $R$ , which is the distinctive property of isobaric functions.

And, *vice versâ*, if  $R$  is homogeneous and isobaric of weight  $w$  and degree  $i$ , then

$$(3a_0\partial_{a_0} + 4a_1\partial_{a_1} + 5a_2\partial_{a_2} + \dots) R = (w + 3i)R = \mu R.$$

Thus homogeneous pure reciprocants are also isobaric and their characteristic is  $3i + w$ . (This property is also true for mixed reciprocants, as we have previously shown.)

2°. Suppose that  $R$  is not homogeneous, but made up of the homogeneous parts

$$R_I, R_{II}, R_{III}, \dots$$

Then, since  $V(R_I + R_{II} + R_{III} + \dots) = 0$

is satisfied identically, it is obvious that

$$VR_I + VR_{II} + VR_{III} + \dots = 0$$

must also be satisfied identically.

But since all the terms are of different degrees, the only way in which this can happen is by making  $VR_I, VR_{II}, VR_{III}, \dots$  separately vanish. Now,  $R_I, R_{II}, R_{III}, \dots$  are by hypothesis *homogeneous* functions of  $a_0, a_1, a_2, \dots$ , and it has just been shown that each of them is annihilated by  $V$ , which has been shown to be a sufficient condition that any homogeneous function of  $a_0, a_1, a_2, \dots$  may be a pure reciprocant. Thus each part  $R_I, R_{II}, R_{III}, \dots$  of  $R$  is a pure reciprocant.

Also, the condition

$$(3a_0\partial_{a_0} + 4a_1\partial_{a_1} + 5a_2\partial_{a_2} + \dots)R = \mu R$$

shows that if  $i_1, w_1; i_2, w_2; i_3, w_3; \dots$  are the deg. weights of  $R_I, R_{II}, R_{III}, \dots$ , we must have

$$3i_1 + w_1 = \mu, 3i_2 + w_2 = \mu, 3i_3 + w_3 = \mu, \dots$$

Thus non-homogeneous pure reciprocants are severable into parts each of which is a homogeneous and isobaric pure reciprocant, the characteristic of each part being equal to the same quantity  $\mu$ , which is the characteristic of the whole.

I will now explain what information concerning the number of pure reciprocants of a given type is afforded by the equation  $VR = 0$ . Let

$$Aa_0^{\lambda_0}a_1^{\lambda_1}a_2^{\lambda_2} \dots a_j^{\lambda_j}$$

be a term of a homogeneous isobaric function (with its full number of terms) of  $a_0, a_1, a_2, \dots, a_j$ , whose degree is  $i$ , extent  $j$ , and weight  $w$ , and which we will call  $R$ .

Then in the entire function there are as many terms as there are solutions in integers of the equations

$$\begin{aligned}\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \dots + \lambda_j &= i, \\ \lambda_1 + 2\lambda_2 + 3\lambda_3 + \dots + j\lambda_j &= w.\end{aligned}$$

In other words, the number of terms in  $R$  is equal to the number of ways in which  $w$  can be made up of  $i$  or fewer parts, none greater than  $j$ . This number will be denoted by  $(w; i, j)$ .

Since the function  $R$  is the sum of every possible term of the form

$$Aa_0^{\lambda_0}a_1^{\lambda_1}\dots a_j^{\lambda_j},$$

each multiplied by an arbitrary constant, the number of these arbitrary constants is also  $(w; i, j)$ .

Now, suppose  $R$  to be a reciprocant; this imposes the condition

$$VR = 0.$$

Consider the effect produced by the operation of any term of

$$V = 4 \left( \frac{a_0^2}{2} \right) \partial_{a_1} + 5a_0a_1\partial_{a_2} + 6 \left( a_0a_2 + \frac{a_1^2}{2} \right) \partial_{a_3} + \dots,$$

say  $\left( a_0a_2 + \frac{a_1^2}{2} \right) \partial_{a_3}$  (rejecting the numerical coefficient 6).

Operating on  $R$  with  $\partial_{a_3}$  decreases its weight by 3 and its degree by 1 unit. The subsequent multiplication by  $a_0a_2 + \frac{a_1^2}{2}$ , on the other hand, increases the weight by 2 and the degree by 2 units. Hence the total effect of  $\left( a_0a_2 + \frac{a_1^2}{2} \right) \partial_{a_3}$  is to increase the degree by 1 and to diminish the weight by 1 unit. The same is evidently true for any other term of  $V$ . Thus the total effect of  $V$  operating on the general homogeneous isobaric function  $R$  of weight  $w$ , degree  $i$ , extent  $j$ , is to change it into another homogeneous isobaric function whose weight, degree and extent are respectively  $w - 1$ ,  $i + 1$ ,  $j$ . Observe that the extent is not altered by the operation of  $V$ .

It is easily seen that the coefficients of  $VR$  are linear functions of the coefficients of  $R$ ; *e. g.* if

$$\begin{aligned}R &= Aa_0^2a_3 + Ba_0a_1a_2 + Ca_1^3, \\ VR &= a_0^3a_2(6A + 2B) + a_0^2a_1^2(3A + 5B + 6C).\end{aligned}$$

Hence the condition  $VR = 0$  gives us  $(w-1; i+1, j)$  linear equations between the  $(w; i, j)$  coefficients of  $R$ ; so that, *assuming that these equations of condition are all independent*, after they have been satisfied the number of arbitrary constants remaining in  $R$  (*i. e.* the number of linearly independent reciprocants of the type  $w; i, j$ ) is equal to

$$(w; i, j) - (w-1; i+1, j)$$

when this difference is positive; but when it is zero or negative there are no reciprocants of the given type.

If, however, any  $r$  of the  $(w-1; i+1, j)$  equations of condition should not be independent of the rest, these equations would be equivalent to  $(w-1; i+1, j) - r$  independent conditions, and therefore the number of linearly independent reciprocants of the type  $w; i, j$  would be

$$(w; i, j) - (w-1; i+1, j) + r.$$

It is therefore certain that this number *cannot be less* than

$$(w; i, j) - (w-1; i+1, j).$$

We shall assume provisionally that  $r = 0$ , or in other words that the above partition formula is exact, instead of merely giving an inferior limit. Though it would be unsafe to rely on its accuracy, no positive grounds for doubting its exactitude have been revealed by calculation.

Such attempts as I have hitherto made to demonstrate the theorem have proved infructuous, but it must be remembered that more than a quarter of a century elapsed between the promulgation of Cayley's analogous theorem and its final establishment by myself on a secure basis of demonstration.

## LECTURE X.

I will commence this lecture with a proof of Capt. MacMahon's theorem that if  $R$  is any pure reciprocant and  $\mu$  its characteristic (*i. e.* its weight added to three times its degree),

$$\left(V^m \frac{d^m}{dx^m}\right) R = 1.2.3 \dots m \{\mu(\mu+2)(\mu+4) \dots (\mu+2m-2)\} (y'')^m R,$$

where  $y''$  may be replaced by either  $2a_0$  or  $a$ , according as the modified or unmodified system of letters is employed.

Instead of a pure reciprocant, let us consider any homogeneous isobaric function  $F$  of degree  $i$  and weight  $w$ ; and (for the sake of simplicity writing  $\partial_x$  for  $\frac{d}{dx}$ ) instead of the operator  $V^m \partial_x^m$  let us consider  $V^m \partial_x^n - \partial_x^n V^m$ . We have identically

$$\begin{aligned} (V^m \partial_x^n - \partial_x^n V^m) F = & V^{m-1} (V \partial_x - \partial_x V) \partial_x^{n-1} F \\ & + V^{m-2} (V \partial_x - \partial_x V) V \partial_x^{n-1} F \\ & + V^{m-3} (V \partial_x - \partial_x V) V^2 \partial_x^{n-1} F \\ & + \dots \dots \dots \\ & + V (V \partial_x - \partial_x V) V^{m-2} \partial_x^{n-1} F \\ & + (V \partial_x - \partial_x V) V^{m-1} \partial_x^{n-1} F \\ & + \partial_x (V^m \partial_x^{n-1} - \partial_x^{n-1} V^m) F. \end{aligned}$$

Now, the operation of  $(V \partial_x - \partial_x V)$  on any homogeneous isobaric function whose characteristic is  $\mu_1$  is equivalent, as we have seen in Lecture VII, to multiplication by  $\mu_1 y''$ ; so that if the characteristics of

$$\partial_x^{n-1} F, \quad V \partial_x^{n-1} F, \quad V^2 \partial_x^{n-1} F, \quad \dots \quad V^{m-1} \partial_x^{n-1} F$$

are  $\mu_1, \mu_2, \mu_3, \dots, \mu_m$ ,

it follows that

$$(V^m \partial_x^n - \partial_x^n V^m) F = (\mu_1 F + \mu_2 + \mu_3 + \dots + \mu_m) y'' V^{m-1} \partial_x^{n-1} F + \partial_x (V^m \partial_x^{n-1} - \partial_x^{n-1} V^m) F.$$

Observe that

$$V^{m-1} (V \partial_x - \partial_x V) \partial_x^{n-1} F = V^{m-1} \mu_1 y'' \partial_x^{n-1} F = \mu_1 y'' V^{m-1} \partial_x^{n-1} F,$$

where the transposition of the  $y''$  is permissible because  $V$  does not act on it; but if  $y''$  were preceded by  $\partial_x$  it could not be similarly transposed.

The numbers  $\mu_1, \mu_2, \mu_3, \dots$  form an arithmetical progression, for each operation of  $V$  increases the degree by unity and diminishes the weight by unity, so that

$$\mu_1 = 3i_1 + w_1 \text{ become } \mu_2 = 3(i_1 + 1) + (w_1 - 1) = \mu_1 + 2.$$

Similarly  $\mu_3 = \mu_1 + 4, \mu_4 = \mu_1 + 6, \dots, \mu_m = \mu_1 + 2m - 2.$

The characteristic of  $F$  being

$$\mu = 3i + w, \text{ that of } \partial_x^{n-1} F \text{ is } \mu_1 = \mu + n - 1;$$

for each operation of  $\partial_x$  leaves the degree unaltered, but adds an unit to the weight; hence

$$\mu_1 + \mu_2 + \mu_3 + \dots + \mu_m = m(\mu + m + n - 2);$$

so that

$$(V^m \partial_x^n - \partial_x^n V^m) F = m(\mu + m + n - 2) y'' V^{m-1} \partial_x^{n-1} F + \partial_x (V^m \partial_x^{n-1} - \partial_x^{n-1} V^m) F. \quad (1)$$

When  $F = R$ , a pure reciprocant, so that  $VR = 0$ , our formula becomes

$$V^m \partial_x^n R = m(\mu + m + n - 2) y'' V^{m-1} \partial_x^{n-1} R + \partial_x V^m \partial_x^{n-1} R. \quad (2)$$

Suppose that in (2)  $m > n$ , then  $V^m \partial_x^n R = 0$ . This is obviously true when  $n = 0$ , and when  $n = 1$ . When  $n = 2$  we find

$$\begin{aligned} V^m \partial_x^2 R &= m(\mu + m) y'' V^{m-1} \partial_x R + \partial_x V^m \partial_x R \\ &= 0 \text{ if } m > 2. \end{aligned}$$

Similarly the case  $n = 3$ ,  $m > 3$  can be made to depend on  $n = 2$ ,  $m > 2$ , and in general each case depends on the one immediately preceding it. Next let  $n = m$  in (2); then, remembering that  $V^m \partial_x^{m-1} R = 0$ , we have

$$V^m \partial_x^m R = m(\mu + 2m - 2) y'' V^{m-1} \partial_x^{m-1} R,$$

from which MacMahon's theorem that

$$V^m \partial_x^m R = 1.2.3 \dots m \{ \mu(\mu + 2)(\mu + 4) \dots (\mu + 2m - 2) \} (y'')^m R$$

is an immediate consequence.

Another special case of Formula (1) is worthy of notice, viz. that in which we take  $n = 1$ , when we obtain the simple formula

$$(V^m \partial_x - \partial_x V^m) F = m(\mu + m - 1) y'' V^{m-1} F. \quad (3)$$

If in this we write  $a_n$  in the place of  $F$ , and (the modified system of letters being used)  $2a_0$  for  $y''$ ,  $\mu$  becomes  $3 + n$ , and we have

$$(V^m \partial_x - \partial_x V^m) a_n = 2m(m + n + 2) a_0 V^{m-1} a_n,$$

or, as it may also be written,

$$\frac{V^m \partial_x a_n}{1.2.3 \dots m} = \frac{\partial_x V^m a_n}{1.2.3 \dots m} + \frac{2(m + n + 2) a_0 V^{m-1} a_n}{1.2.3 \dots (m-1)}. \quad (4)$$

Mr. Hammond remarks that this last formula may be used to prove the theorem

$$\alpha_n = -t^{-n-3} \left( e^{-\frac{r}{t}} \right) a_n,$$

which was given without proof in Lecture II. Assuming that

$$\alpha_n = -t^{-n-3} a_n + t^{-n-4} V a_n - t^{-n-5} \frac{V^2 a_n}{1.2} + \dots,$$

we have to prove that the theorem is also true when  $n$  is increased by unity.

Differentiating both sides of the assumed identity with respect to  $x$ , we find

$$\begin{aligned} \partial_x \alpha_n &= \partial_x \left( -t^{-n-3} a_n + t^{-n-4} V a_n - t^{-n-5} \frac{V^2 a_n}{1.2} + \dots \right) \\ &= -t^{-n-3} \partial_x a_n + t^{-n-4} \{ \partial_x V a_n + 2(n+3) a_0 a_n \} \\ &\quad - t^{-n-5} \left\{ \frac{\partial_x V^2 a_n}{1.2} + 2(n+4) a_0 V a_n \right\} \\ &\quad + \dots; \end{aligned}$$

the general term being

$$(-)^{m+1}t^{-n-m-3}\left\{\frac{\partial_x V^m a_n}{1.2.3\dots m} + \frac{2(m+n+2)a_0 V^{m-1}a_n}{1.2.3\dots(m-1)}\right\}$$

which, by means of (4), reduces to

$$(-)^{m+1}t^{-n-m-3}\frac{V^m\partial_x a_n}{1.2.3\dots m}.$$

Hence

$$\partial_x a_n = -t^{-n-3}\partial_x a_n + t^{-n-4}V\partial_x a_n - t^{-n-5}\frac{V^2\partial_x a_n}{1.2} + \dots,$$

or, more concisely,  $\partial_x a_n = -t^{-n-3}\left(e^{-\frac{V}{t}}\right)\partial_x a_n$ .

But  $\partial_x a_n = (n+3)a_{n+1}$ , and  $\partial_x a_n = t\partial_y a_n = (n+3)ta_{n+1}$ ,

and therefore  $(n+3)ta_{n+1} = -(n+3)t^{-n-3}\left(e^{-\frac{V}{t}}\right)a_{n+1}$ ,

or  $a_{n+1} = -t^{-n-4}\left(e^{-\frac{V}{t}}\right)a_{n+1}$ .

The theorem is easily seen to be true, for  $n = 0, 1, 2$ , and is thus proved to be true universally.

I will now return to the point at which I left off in my previous lecture. We saw that the exactitude of the formula

$$(w; i, j) - (w-1; i+1, j)$$

for the number of pure reciprocants of the type  $w; i, j$  could not be inferred with certainty unless we were able to prove that the  $(w-1; i+1, j)$  linear equations between the coefficients of  $R$ , found by equating  $VR$  to zero, were all of them independent. A similar difficulty presents itself in the proof of the corresponding formula  $(w; i, j) - (w-1; i, j)$  in the invariantive theory; but in that case I succeeded in making out a proof of the independence of the equations of condition founded on the fact that  $\Omega^m O^m I$  is a numerical multiple of  $I$ , where  $I$  is any invariant, and  $\Omega, O$  are the well-known operators

$$\begin{aligned} & a_0\partial_{a_1} + 2a_1\partial_{a_2} + 3a_2\partial_{a_3} + \dots + ja_{j-1}\partial_{a_j} \\ & a_j\partial_{a_{j-1}} + 2a_{j-1}\partial_{a_{j-2}} + 3a_{j-2}\partial_{a_{j-3}} + \dots + ja_1\partial_{a_0}. \end{aligned}$$

I have since discovered a second proof of the theorem for invariants which, though very interesting, is less simple than my first; but neither of these methods can be extended to the case of reciprocants.

It was suggested by Capt. MacMahon that the fact that  $V^m\partial_x^m R$  is a numerical multiple of  $a^m R$  ought to lead to a proof of the theorem for reciprocants

similar to that obtained for invariants by my first method, alluded to above, but this I find is not the case; and indeed it is capable of being shown *a priori* that it cannot lead to a proof. One great distinction between the two theories, which is fatal to the success of the proposed method, is well worthy of notice.

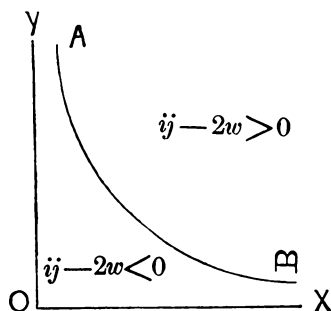
If  $(w; i, j) - (w - 1; i, j) \geq 0$  (I shall sometimes call this positive), then  $(w'; i, j) - (w' - 1; i, j) \geq 0$  for all values of  $w'$  less than  $w$ ;

the condition that this difference, say  $\Delta(w; i, j)$  shall be positive being simply that  $ij - 2w$  is positive (*i. e.*  $ij - 2w \geq 0$ ). This is not the case with the difference  $(w; i, j) - (w - 1; i + 1, j)$ , say  $E(w; i, j)$ ; it by no means follows that if this is positive for a given value of  $w$  ( $i, j$  being kept constant), it will be so for any inferior value of  $w$ .

We may illustrate geometrically the condition  $ij - 2w \geq 0$ , which holds when  $\Delta(w; i, j)$  is non-negative.

Let  $(i, j)$  be co-ordinates of a point in a plane and draw the positive branch of the rectangular hyperbola

$$ij - 2w = 0.$$



Then,  $ij - 2w < 0$  for all points in the area  $YOXBA$  between the curve and its asymptotes; but for points on the curve  $AB$ ,

$$ij - 2w = 0,$$

and for all points of the infinite area on the side of  $AB$  remote from the origin,

$$ij - 2w > 0.$$

Thus, for all points which lie either on or beyond the curve  $AB$ ,

$$\Delta(w; i, j) \text{ is non-negative,}$$

and for all points between the curve and the asymptotes

$$\Delta(w; i, j) \text{ is non-positive.}$$

We have here considered  $w$  as constant and  $i, j$  as variable, but in the case where all three are variable we should have to consider the hyperbolic paraboloid

$$ij - 2w = 0,$$

of which the curve  $AB$  is a section, by the plane  $w = \text{const.}$ ; and the condition of  $\Delta(w; i, j)$  being non-negative or non-positive depends on the variable point  $(i, j, w)$  lying in the one case on or beyond the surface, and in the other between the surface and the planes of reference.

The function of  $i, j, w$ , whose positive or negative sign determines in like manner that of  $E(w; i, j)$ , cannot be linear in  $w$ . What its form is, or whether it is an Algebraical or Transcendental function, no one at present can say. Indeed, except for the light shed on the subject by the Algebraical Theory of Invariants, it would have been exceedingly difficult (as I know from vain efforts made by myself and others in Baltimore) to prove the much simpler theorem that  $\Delta(w; i, j)$  is positive (*i. e.* non-negative) when  $ij - 2w$  is so. It amounts to the assertion that the coefficient of  $\alpha^t x^w$  in the expansion of

$$\frac{1 - x}{(1 - \alpha)(1 - \alpha x)(1 - \alpha x^2) \dots (1 - \alpha x^j)}$$

is always non-negative, provided that  $ij - 2w$  is non-negative.

This is a theorem of great importance in the ordinary Theory of Invariants, and may be seen to be a consequence of the fact, which I have proved, that (using  $[w; i, j]$  to denote a function of the type  $w; i, j$  having its full number of arbitrary coefficients) there are no linear connections between the coefficients of  $\Omega[w; i, j]$  when  $ij - 2w = > 0$ ; but no one, as far as I know, has ever found a *direct* proof of it.

Viewing the connection between the two theories of Invariants and Reciprocants, I think it desirable to recapitulate with some improvements the proof, given in the Phil. Mag. for March, 1878, of the theorem that the number of linearly independent invariants of the type  $w; i, j$  is exactly  $\Delta(w; i, j)$  when this quantity is positive, and exactly zero when it is 0 or negative.

As regards reciprocants, at present we can only say that the number of linearly independent ones of the type  $w; i, j$  is never less than  $E(w; i, j)$ , leaving to some gifted member of the class to prove or disprove that the first is always exactly equal to the second. The *exact* theorem to be proved in the theory of invariants is as follows:

If  $ij - 2w = > 0$ , the number of linearly independent invariants of the type  $w; i, j$  is  $\Delta(w; i, j)$ .

If  $ij - 2w < 0$ , the number of such invariants is zero; *i. e.* there are none. The proof is made to depend on the properties of

$$\Omega = a_0 \partial_{a_1} + 2a_1 \partial_{a_2} + 3a_2 \partial_{a_3} + \dots + ja_{j-1} \partial_{a_j}$$

and of 
$$O = a_j \partial_{a_{j-1}} + 2a_{j-1} \partial_{a_{j-2}} + 3a_{j-2} \partial_{a_{j-3}} + \dots + ja_1 \partial_{a_0}.$$

If  $U$  be any homogeneous isobaric function of degree  $i$  and weight  $w$  in the letters  $a_0, a_1, a_2, \dots, a_j$ , it is easy to prove that

$$(\Omega O - O\Omega) U = (ij - 2w) U,$$

and consequently, if  $U$  is an invariant  $I$ , so that  $\Omega I = 0$ ,

$$\Omega O I = (ij - 2w) I.$$

I call  $ij - 2w$  the *excess* and denote it by  $\eta$ , and shall first show that if  $\eta$  is negative  $I = 0$ ; *i. e.* there exists no invariant with a negative excess. This will prove that when  $\Delta(w; i, j)$  is negative, *i. e.* when  $(w - 1; i, j) > (w; i, j)$ , the number of independent functions of the coefficients of  $[w; i, j]$  which appear in  $\Omega [w; i, j]$  is exactly equal to  $(w; i, j)$ , which is the number of the coefficients themselves. Clearly it cannot be greater; for, no matter what the number of linear functions of  $n$  quantities may be, only  $n$  at the utmost can be independent; there might be fewer, there cannot possibly be more. The complete theorem is that the number of independent coefficients in  $\Omega [w; i, j]$  is the *subdominant* of two numbers: one the number of terms of the type  $w; i, j$ , the other the number of terms of the type  $w - 1; i, j$ .

N. B.—That one of two numbers which is not greater than the other is called the subdominant.